Maximum subsets in Euclidean position in Euclidean 2-orbifolds and the sphere

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Abstract

Intuitively, a set of sites on a surface is in Euclidean position, if points are so close each other that planar algorithms can be easily adapted in order to solve classical problems of Computational Geometry. In this work we focus in a very relevant class of metric surfaces, the Euclidean 2-orbifolds in addition to the sphere. To seek for maximum sets (in terms of cardinal) in Euclidean position of a given set of sites is, in most of these surfaces, equivalent to compute the maximum depth of an arrangement of convex sets determined by the geometry of the surface. We present algorithms for finding either one or all the maximum subsets as well as the number of such subsets and their minimum cardinal and show that this problem is equivalent to the search of the maximum clique for a natural class of geometric graphs generated from these surfaces.

1 Introduction. Euclidean position

Most people along the human history have believed in a *flat earth*. Even nowadays there exit persons that still hold the flatness of the world. This is because most of our daily experience takes place in a restricted region of a sphere-like surface, so that there are no significant errors if it is considered as a plane. This idea can be easily extended to the Computational Geometry context, where in multiple applications it is assumed that if a given data set is constrained to a small portion of a surface it presents a planar behavior.

The notion of Euclidean position was introduced in [7] for the sphere, the cylinder, the cone, and the torus, and it was extended in [4] to a wider class of surfaces, the Euclidean 2-orbifods, where methods for determining whether or not a point set is in Euclidean position are developed (see first column in Table 1). In this paper we extend those previous works and focus on finding the subsets with highest cardinal in Euclidean position either on an Euclidean 2-orbifold or on the sphere.

As it is known, any Euclidean 2-orbifolds is obtained as a quotient space $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma \simeq S$, being φ the quotient map and Γ a discrete group of planar motions, and where an equivalent class (the *orbit*) of a point of $\mathbb{R}^2$ is given by all its images by elements of Γ. The four locally Euclidean surfaces (the cylinder, the twisted cylinder, the torus and the Klein bottle) and others well-known surfaces as the Moebius strip or the projective plane are 2-orbifolds. A more complete study of them can be found in [8].

Given two points, X and Y, the geodesics joining them in the quotient metric correspond to straight-line segments matching one point of the orbit of X with all the ones of the orbit of Y. The shortest of these line segments will be called the *segment* between X and Y. On the sphere,

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the *segment* corresponds to the shortest geodesic joining the points. The distance between X and Y is the length of the segment joining them.

To get a simple representation of an Euclidean 2-orbifold, a very useful tool is the *fundamental domain*: a closed region in the plane containing one element of each orbit, that is unique except for the points of the boundary (*double points*). If we delete all double points of a fundamental domain and consider its image by φ, we obtain what we call a *fundamental region*.

A set of sites P either on a 2-orbifold or the sphere is said to be in *Euclidean position* if there exists a fundamental region so that all segments joining points of P (the clique generated by P) are contained inside it (a hemisphere in the case of the sphere).

A subset Q of P is a *maximum subset of P for the Euclidean position* (MSEP for short) if it is in Euclidean position and there is no other higher cardinal subset having that property. In this work we develop several methods to find either *one* or *all* the MSEPs of a given set of sites both on the Euclidean 2-orbifolds and the sphere and determine how large *all* means. Complexities of these procedures are summarized in Table 1 together with the minimum number of points of P that can be assured to be in Euclidean position on each surface.

Given a set P on an Euclidean 2-orbifold or the sphere, the *segment graph* of P is the geometric graph having the points of its orbit as vertices, and as edges all the possible segments, that coincides with the image by φ^{-1} of the complete graph with nodes in P. It is easy to realize the following result:

Lemma 1 *$Q \subset P$ is in Euclidean position if and only if the connected components of the segment graph of Q are cliques.*

This turns our problem to the searching of cliques in segment graphs, a problem that is known to be NP-hard for general graphs [3, 6]. However, for segment graphs on the sphere and most of 2-orbifolds (the ones without glide reflections) we prove that this can be solved in polynomial time. This is due to the geometric properties of these surfaces which allow to transform this problem to the one of computing the region of maximum depth in an arrangement of convex polygons (or maximum circles in the sphere).

We will make use of known algorithms for computing the maximum depth of an arrangement of convex polygons [5, 2] as well as the ones for determining the halfspace depth of a point in both the plane [1] and the space [9]. This reasoning does not work properly in surfaces with glide reflections, where no polynomial time algorithms have been found at once. In fact, sets with $O(2^n)$ MSEPs can be constructed in these surfaces.

2 Surfaces without glide reflections

It is known a set of sites P on the cone, the cylinder, the torus, or the sphere is in Euclidean position if and only if it is contained in certain sets whose shape depend on the surface considered (between opposite generatrices of the cylinder or the cone; a quadrant -the region between two opposite parallels and two opposite meridians- of the flat torus; or an hemisphere of the sphere). By taking the counter image by the quotient map of these sets in the plane, it can be proven [7, 4] that P is in Euclidean position if and only if its orbit is contained in some particular convex sets (according on the surface considered) in the plane.

Thus, the searching of MSEPs turns to the optimal placement problem of a certain convex set containing the maximum number of points of $\varphi^{-1}(P)$. This can be solved by computing the maximum depth of the arrangement of sets centered at the points of the orbit of P, so the next result holds:

Theorem 1 *Let $S = \mathbb{R}^2/\Gamma$ be an Euclidean 2-orbifold (respectively the sphere), with Γ a discrete group of motions containing no glide reflections, and a set of sites P on S. Then it can be constructed an arrangement of convex polygons (respect. maximum circles) such as any region of the arrangement with maximum depth is associated to a MSEP of P.*

The region with maximum depth in an arrangement of n convex sets can be computed in $O(nk \log n)$ time, where k is the depth of the arrangement, for translations of a fixed convex set in the plane [2]; or, as it is noted in [5], by modifying the algorithm given in [10], that provides a computational time of $O(n^{d/2} \log n)$ for an arrangement of isotetic hipercubes in $\mathbb{R}^d$. These algorithms together with Theorem 1 give rise to the following assertion:

Corolary 1 *A maximal clique in a segment graph generated from a set in a 2-orbifold without glide reflection or the sphere can be found in polynomial time.*

The convex polygons cited in Theorem 1 depend on the 2-orbifold considered, and it is

- a strip for sets given on the cylinder,
- an angular sector with vertex on the center of the rotation for the cone,
- an isotetic rectangle for the flat torus (generated from two orthogonal translations) and the pillow-like surfaces (generated from rotations),
- a hexagon for the skew torus (non-orthogonal translations), or
- an isotetic cube for the Pillow (generated by two π radians rotations and a translation or by four π radians rotations).

The time needed to compute the maximum depth of arrangements of such sets are summarized in second column of Table 1. The optimum for the cylinder and the cone are due to the MSEP search in these surfaces can be connected to the halspace depth problem; that is, to determine the halfspace, whose boundary contains a fixed point (the center of the circumference in our case), having less points of a given set [1]. This also assures the optimum for the flat torus and the pillow-like surfaces.

Note as the angle between the generating translations in the skew torus cause a change of the polygon considered that increases the computing time. This $O(n^{1.5} \log n)$ time improves the $O(nk \log n)$ time given in [2] for arrangement whose depth k is greater than $\sqrt{n}$.

In spite of what happen on the 2-orbifolds, the searching of a MSEP on the sphere can be done by computing the maximum depth region of an arrangement of maximum circles on the sphere itself. This problem is equivalent either to determine the subset of highest cardinal of a point set in $\mathbb{R}^3$ whose convex hull does not contain a fixed point (the center of the sphere) or to compute the halfspace depth of the center, and improves the $O(n^2 \log n)$ time given in [9].

Theorem 2 *Given a set of n sites on $\mathbb{R}^3$, the halspace depth of a given point can be computed in $O(n^2)$ time.*

Finally, in Table 1 are also listed the number of MSEPs on each surface and the time necessary to find all of them, together with their minimum cardinal.

3 Surfaces with glide reflections

If a glide reflection is involved, there is not a unique region G such as a set is in Euclidean position if and only if it is contained in G. 2-orbifolds generated by this motion are non-orientables, and it includes the Moebius strip, the Klein bottle or the projective plane. In this surfaces it is not possible to make use of Theorem 1. If fact, opposite the other 2-orbifolds, there can be constructed sets with $O(2^n)$ MSEPs, and $\Theta(2^n)$ time is required to report all of them. We are actually working in determining if it is possible to find *one* MSEPs (and as a consequence, a clique in the segment graph) in polynomial time.

4 Conclusions and open problems

Problem we have worked on are summarized in Table 1; computing either one or all MSEPs in Euclidean position and giving bounds for the number of such sets and for the minimum number of points that take part of any of them.

	Determine [4]	Find a max. set	Num. of max. sets	Find all max. sets	Min. num. of points
Cylinder	$\Theta(n)$	$\Theta(n \log n)$	$O(n)$	$\Theta(n \log n)$	$n/2$
Cone	$\Theta(n)$	$\Theta(n \log n)$	$O(n)$	$\Theta(n \log n)$	$n/2$
Torus	$\Theta(n)$	$\Theta(n \log n)$	$O(n^2)$	$\Theta(n^2)$	$n/4$
Skew Torus	$\Theta(n)$	$O(n^{1.5} \log n)$	$O(n^2)$	$\Theta(n^2)$	$n/4$
Pillow	$\Theta(n)$	$O(n^{1.5} \log n)$	$O(n^3)$	$\Theta(n^3)$	$n/2$
Pillow-like surfaces	$\Theta(n)$	$\Theta(n \log n)$	$O(n^2)$	$\Theta(n^2)$	$n/2$
Surfaces with glide reflections	$O(n \log n)$	?	$O(2^n)$	$\Theta(2^n)$	$n/4$
Sphere	$\Theta(n)$	$O(n^2)$	$O(n^2)$	$\Theta(n^2)$	$n/2$

Table 1: Scheme of problems and the current cost of solutions.

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