The anchored Voronoi diagram: static and dynamic versions and applications

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Abstract

Given a set S of n points in the plane and a fixed point o, we introduce the Voronoi diagram of S anchored at o. It will be defined as an abstract Voronoi diagram that uses as bisectors the following curves. For each pair of points p, q in S, the bisecting curve between p and q is the locus of points x in the plane such that the line segment $\overline{ox}$ is equidistant to both p and q. We show that those bisectors have nice properties and, therefore, this new structure can be computed in $O(n \log n)$ time and $O(n)$ space. Also, under a slightly different model of computation, we prove that the dynamic version of this diagram can be built in $O(n^2 \lambda_{6s+2}(n))$ time complexity, where s is a constant depending on the function that describes the motion of the points. Both static and dynamic diagrams can be used for solving maximin location problems, where the goal is the placement of a line segment connecting two fixed curves.

1 Introduction

Given a set of n sites in a continuous space, the subdivision of the space into regions, one per site, according to some influence criterion is a central topic in Computational Geometry and it has been applied to many fields of science. The standard name for this geometric structure is due to Voronoi, who proposed the first formalization. Originally, this structure was used for characterizing regions of proximity for the sites. Since then, mMany extensions and generalizations have been proposed (see the surveys [1, 5, 9]). Also, other general approachs have been introduced [4, 8] where the concepts of site or distance functions are not explicitly used. In this paper, we introduce an abstract Voronoi diagram in the sense of [8], *the anchored Voronoi diagram*. In section 2, we formally define this structure, give some properties and show how to compute it; we also give an application to a facility location problem. In section 3, we deal with the dynamic version of the anchored diagram; in particular, we discuss the topological matters that lead to its construction and, finally, we show how to apply this structure to solving some maximin problems. Those problems consist of finding the bridge that connects to curves so that the minimum distance from the bridge to a given point set is maximized. Concluding remarks of the paper are put forward in Section 4 .

2 The anchored Voronoi diagram

2.1 Definition and properties

Given a set S of n points in the plane, the Euclidean distance between two points p and q will be denoted by $d(p, q)$. We define an *anchored* segment as a line segment when the initial point is fixed. Without loss of generality, we will consider the anchor to be the origin. Finally, the distance between a point p and an anchored segment connecting o with a point $x \in \mathbb{R}$ will be defined as $d(p, \overline{ox}) := \min\{d(p, q) : q \in \overline{ox}\}$.

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For any two different points p, q in S, a *bisecting curve* $L(p, q)$ is defined as the locus of points x in the plane such that the line segment $\overline{ox}$ is equidistant to both p and q, that is, $L(p, q) = \{x \in I\!R^2 : d(p, \overline{ox}) = d(q, \overline{ox})\}$. An exhaustive study of the properties and the shape of $L(p, q)$ have been carried out in [2]. $L(p, q)$ is homeomorphic to a line and dissects the plane into two open domains $D(p, q)$ and $D(q, p)$ having $L(p, q)$ as boundary. We define the *Anchored Voronoi Region* $AVR(p, S)$ to be the intersection of the domains $D(p, q)$, where $q \in S \setminus \{p\}$. Then the *Anchored Voronoi Diagram* $AVD(S)$ of the bisecting curves $L(p, q)$ is defined as the union of all boundaries of at least two Voronoi region have in common. In Figure 1, all types of bisecting curves are shown. Note that in the case (a.3) $L(p, q)$ includes a region. In order to simplify the discussion and for this degenerate situation,we will take $L(p, q) = \{x \in I\!R^2 \mid d(p, x) = d(q, x)\}$ as the bisector of the line segment $\overline{pq}$.

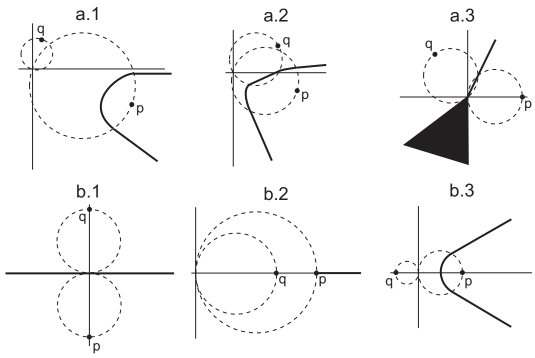

Figure 1: The locus $L(p, q)$.

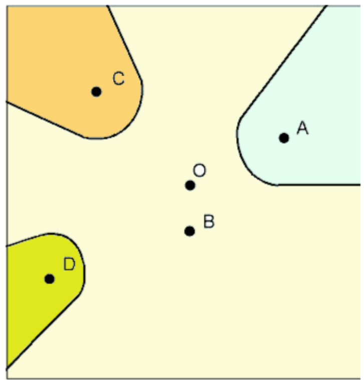

Figure 2: $AVD(S)$ for four points.

An edge can be composed into pieces which are either half-lines, or line segments, or arcs of a curve of degree four or arcs of a circle (refer to Figure 2). A vertex of $AVD(S)$ (defined by at most three points) can either be a point or a half-line or an arc of circle. Note that vertices are defined as the intersections of the Voronoi edges.

2.2 Computation

In order to compute the $AVD(S)$, the divide & conquer approach given in [8] can be used. The system $\mathcal{L} = \{L(p, q) : p, q \in S, p \neq q\}$ is called *admissible* iff for each subset S' of S of size at least 3 the following conditions are fulfilled: (1) the Voronoi regions are path-connected; (2) each point of the plane lies in a Voronoi region or on the Voronoi diagram; (3) the intersection of two bisecting curves only consists of finitely many components. By using non-trivial geometrical properties, we have proved the following results.

Lemma 2.1 *The set of locus $\mathcal{L}$ is an admissible system.*

Theorem 2.1 *The Anchored Voronoi Diagram of a set of point S in the plane can be constructed in $O(n \log n)$ time and $O(n)$ space.*

2.3 Application

We next show how to use the $AVD(S)$ as a data structure for computing a solution for a location maximin problem. The *obnoxious anchored bridge* problem is stated as follows:

OABP: *Let S be a set of n points in $I\!R^2 \setminus \{o\}$ and let $\mathcal{C}$ be a curve (typically, in most applications, an algebraic curve of constant degree). Compute a line segment connecting o with a point x on $\mathcal{C}$ for which $\min_{p \in S} d(p, \overline{ox})$ is maximized.*

Typically, in most applications, curve $\mathcal{C}$ will be an algebraic curve of constant degree, a trigonometric function or similar.The following results solve this problem.

Lemma 2.2 *There exists a point x^* which is the intersection between the curve $\mathcal{C}$ and the structure $AVD(S)$ such that the segment $\overline{ox^*}$ is a solution for the problem OABP.*

Theorem 2.2 *Once the $AVD(S)$ is given, the problem OABP can be solved in linear time and space.*

At this point, we should note that there are certain operations here that exceed the power of the usual real RAM model. The model of computation should be augmented with the pertinent primitives as neccesary.

3 The dynamic anchored Voronoi diagram

We are given a finite set of $n \geq 3$ points $S = \{p_1, \ldots, p_n\}$ each moving along a polynomial trajectory of maximum degree s, for some constant s. Let $p_i(t)$ denote the position of point p_i at time t. We further assume that the points move without collisions.

Definition 3.1 *Given an anchored segment S of length l and $\varepsilon \geq 0$, the locus of points that are at distance ε from S is called an anchored hippodrome centered at S of radius ε.*

In our context, we consider the points in general position when no four points are *co-hippodromal*, in other words, there not exists a hippodrome with four points on the boundary. We study how the structure $AVD(S(t))$ changes with time. Similarly to the ordinary dynamic Voronoi diagram [6], $AVD(S)$ changes continuously but its combinatorial structure only changes at critical values of t. We will call the dual graph of $AVD(S(t))$ *the anchored graph* $AG(S(t))$.

3.1 Topological changes

In order to obtain a bound of the number of changes in $AG(S(t))$ we describe how an edge can be removed or added to the graph as the points move. In the following, we characterize these elementary changes.

It follows from the definition of the anchored dual graph that there is an edge between two points if and only if there exists an empty hippodrome that passes through those points (The converse is also true.) Let $p_i(t)$, $p_j(t)$ be two points in $S(t)$ and let be given a hippodrome that have them on its boundary. This hippodrome determines a unique line segment, one of whose endpoints is the origin and the other is a point $x_{ij}(t)$. Of course, $x_{ij}(t)$ lies on the Voronoi edge contained in the bisector curve $L(p_i(t), p_j(t))$. Let $d_{ij}(t)$ denote the distance from point $p_i(t)$ to line segment $\overline{ox_{ij}(t)}$. When a point $p_k(t)$ enters into an empty hippodrome given by points $p_i(t)$, $p_j(t)$, then a combinatorial change takes place. Such change will correspond to an intersection between function $d_{ij}(t)$ and another function $d_{ik}(t)$ or $d_{jk}(t)$. What really matters here is when the first point that enters into the empty hippodrome, which results in only considering the lower enveloppe of the functions $\{d_{ij}(t), i \neq j\}$. By examining those intersections, we can give an upper bound on the number of topological changes.

We can see that any pair of functions $\{d_{ij}(t), i \neq j\}$ intersects at most $6s$ times. Hence, the number of breakpoints of the lower envelope of the distance functions is $O(\lambda_{6s+2}(n))$. By repeating this argument for all pair of points in $S(t)$, we obtain the desired upper bound. We thus conclude with the following theorem.

Theorem 3.1 *The number of topological changes of the combinatorial structure of $AVD(S(t))$, when each point in S moves along a trajectory defined by polynomial of maximum degree s, is $O(n^2\lambda_{6s+2}(n))$.*

3.2 Computing the diagram

The topological structure of an anchored Voronoi diagram under continuous motions of the points in S can be maintained dynamically. By using a similar approach to those in [3, 7], we are able to update the changes in $O(\log n)$ time.

Theorem 3.2 *The dynamic $AVD(S)$ can be constructed in $O(n^2\lambda_{6s+2}(n)\log n)$ time.*

3.3 Application

With the dynamic anchored Voronoi diagram we can solve a general problem for locating an obnoxious bridge. Consider the problem of Section 2.3 but we suppose the anchor point o can be moved through a polynomial trajectory. The problem now is:

OBP: *Let S be a set of n points in $\mathbb{R}^2$, $\mathcal{P}$ be a polynomial curve and $\mathcal{C}$ a curve. Compute a line segment l connecting $\mathcal{P}$ with $\mathcal{C}$ for which $\min_{p\in S} d(p,l)$ is maximized.*

Observe that we can solve this problem by fixing an endpoint of the segment l on a point of $\mathcal{P}$ and moving the points in S along such a trajectory. Recall that the $OABP$ problem can be solved by finding the Voronoi vertices for which the segment is the center of the largest empty hippodrome. Then, as t varies, we maintain for each Voronoi vertex the starting time t_0 at which appears and also the time t_1 at which disappears. Between t_0 and t_1, we compute when the width of the corresponding hippodrome is maximized. Finally, by keeping track of those maximum values, the general problem OBP can be solve within the time obtained for the computation of the dynamic anchored Voronoi diagram.

4 Conclusion

We have introduced in this paper the anchored Voronoi diagram as an abstract Voronoi diagram. The bisecting curves are induced by the distance to a line segment anchored at the origin. Also, for the dynamic version we have found an upper bound based in a Davenport-Schinzel argument. We have finished by showing an application, namely, solving a maxmin bridge problem.

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