# Approximating planar subdivisions and generalized Voronoi diagrams from random sections

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**Abstract**

We present an algorithm for constructing from a set of sampled sections a piecewise-linear approximation of an unknown planar subdivision, including generalized Voronoi diagrams as outstanding example. The input of the algorithm is a set of lines uniformly distributed over the theoretical subdivision. For each input line, the ordered set of sections in which the line and the planar subdivision intersect is given or computed. The algorithm outputs a triangulation from which the approximation of the unknown subdivision, both in the topological and the metrical sense, can easily be extracted. The correctness of the algorithm and the evaluation of its time complexity follow from results of Integral Geometry and Geometric Probability.

## 1 Introduction

We present an algorithm that allows to construct a piecewise-linear approximation of an unknown planar subdivision from a set of sampled line-sections. The input of our algorithm is a sufficiently "dense" set of lines uniformly distributed over a bounding box together with the ordered set of sections in which each line intersects the theoretical planar subdivision. We want to remark that, in the context of integral geometry and geometric probability, uniformly distributed means that the probability that a line intersects a piece of the boundary of and edge, independent of its location and orientation, is proportional to its length [17]. The algorithm outputs a triangulation from which a planar subdivision approximating the unknown planar subdivision can easily be extracted. The input lines are sequentially processed and the method is progressive, in the sense that for each line we obtain a new approximation from the previous one.

Some of the ideas used in our algorithm extend previous work on reconstructing planar shapes from random sections as a problem related to curve reconstruction [6]. Different cases of the curve reconstruction problem have been treated: uniformly or non-uniformly sampled points, closed or open curves, smooth or non-smooth curves and many algorithms have been proposed for all these cases [2, 3, 4, 7, 8, 9, 10, 11, 13, 14, 15]. The main difference between these algorithms and ours is that all them compute the reconstruction off-line, while our algorithm works on-line. As mentioned above, the fact that the random lines are sequentially processed as they arrive allows also progressive refinement that may be tuned by the user.

The method can be applied to the construction of planar subdivisions, and in particular to generalized Voronoi diagrams, that correspond to two-dimensional scenarios for which the whole

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structure is difficult to obtain, but such that its intersection with any given line is easy to compute or to obtain with some device.

Voronoi diagrams are a fundamental structure in computational geometry, both for the rich theory they embody as for the impressive variety and number of applications they have. Many variants have been considered: by taking sites of different shape or nature, associating weights to the sites, changing the underlying metrics, or using individualized distance functions for the sites. Classic and generalized Voronoi diagrams are described in the surveys [1, 16].

The algorithms designed for computing exact generalized planar Voronoi diagrams often have numerical robustness problems and are time-consuming due to the numerous high precision calculations that are required. However in some applications (like motion planning or geographic map simplification) the computation of an approximated Voronoi diagram within a predetermined precision is sufficient, and several algorithms have been proposed for the approximation of Voronoi diagrams [5, 12, 18, 19]. We present an algorithm for approximating generalized planar Voronoi diagrams for different site shapes (points, line-segments, curve-arc segments, ...) and different distance functions (Euclidean metrics, convex distance functions, ...). The algorithm is robust and fast (in terms of running time). Moreover it is very general: not all the sites must be homogeneous in shape or have associated the same distance function, and there are no restrictions on the connectivity of the Voronoi regions –for a single site they may have several components–, the degree of the Voronoi vertices, or the dimensionality of the bisectors.

# 2 Sketch of the method

We assume that the planar subdivision $\mathcal{P}$ to be approximated is contained in a tight axis-parallel bounding-box $K$. The algorithm takes as input a set $L$ of $m$ lines uniformly distributed over $K$. For each line $l$ of $L$ we have the ordered set $S(l)$ of intersections between $l$ and the regions of $\mathcal{P}$. The algorithm outputs a triangulation $T(\mathcal{P})$ of $K$ represented by a DCEL structure. Each triangle of the $T(\mathcal{P})$ is assigned to one of the regions of $\mathcal{P}$ or to an auxiliary region that we call background. From this triangulation it can easily be obtained, in cost linear with respect to the number of triangles, the piecewise-linear approximation $A(\mathcal{P})$ of $\mathcal{P}$.

The main part of the algorithm processes the sections of the lines of $L$ sequentially. When a new section is considered, the triangles that contain the endpoints of the section are subdivided, and a region determined by a subset of the triangles crossed by the section is retriangulated in order to connect two vertices by an edge. The DCEL structure is actualized properly. To describe the general idea followed to design the algorithm, we explain the particular case of the five sections of Figure 1. First we introduce section $a$ as an edge. Since section $b$ intersects section $a$, a triangulated quadrilateral that connects the endpoints of section $b$ with the endpoints of section $a$ is created. As section $c$ do not intersect the quadrilateral, then it is inserted as an edge. Since section $d$ intersects section $c$, a second triangulated quadrilateral is also created. Section $e$ intersects the two quadrilaterals, then the endpoints of the section are connected with the quadrilaterals, and the edges of the quadrilaterals crossed by the section are also connected. In order to guarantee the desired cost of the algorithm, we maintain the length of the new edges produced by the subdivision of the triangles inversely proportional to the number of processed lines.

We can prove, using results of Integral Geometry and Geometric Probability, that by taking the number of lines $m$ of $L$ large enough the piecewise-linear subdivision $A(\mathcal{P})$ obtained with our algorithm tends to the planar subdivision $\mathcal{P}$ in both the topological and the metrical sense, and we can show also that the mean computational cost of the algorithm is $O(m \log m)$.

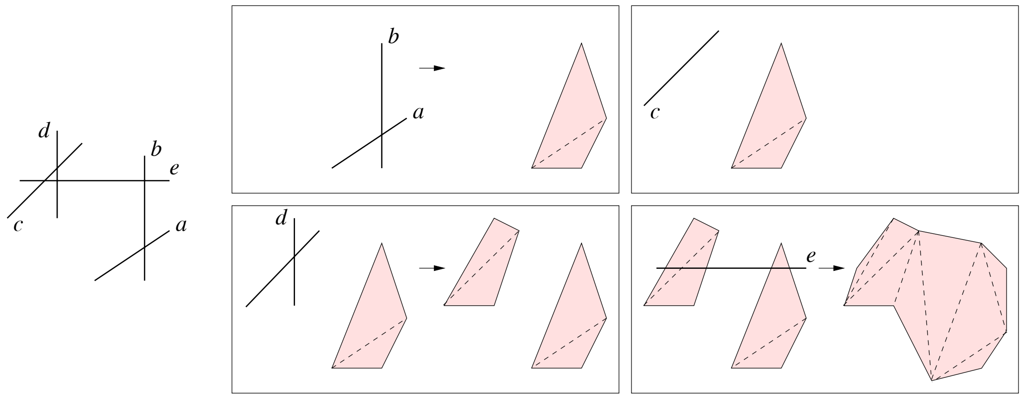


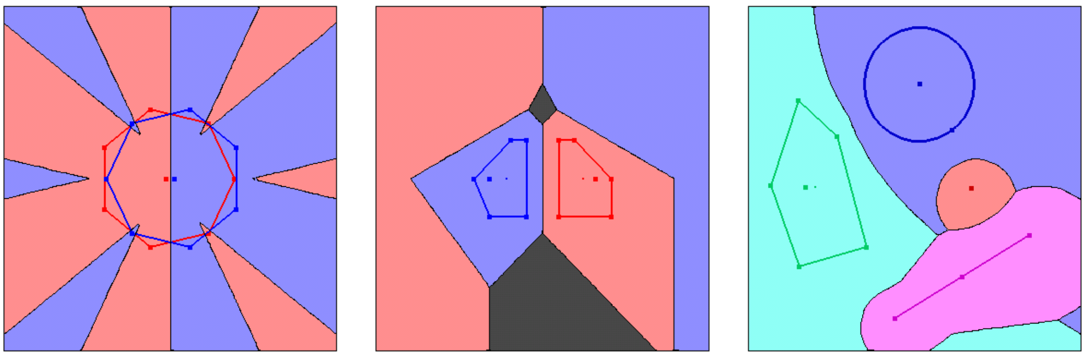

Figure 1: *Triangulation process*

# 3 Voronoi diagram approximation

Our algorithm can be used in order to approximate generalized planar Voronoi diagrams for different site shapes (points, line-segments, curve-arc segments, ...) and different distance functions (Euclidean metrics, convex distance functions, ...). The resulting approximation algorithm is very general: not all the sites must be homogeneous in shape or have associated the same distance function, and there are no restrictions on the connectivity of the Voronoi regions, which for a single site may have several components, the degree of the Voronoi vertices or the dimensionality of the bisectors.

The total cost of approximating the generalized Voronoi diagram of $n$ sites inside a bounding-box $K$, using $m$ lines uniformly distributed on $K$, is $O(mn \log n + m \log m)$.

We have implemented both the general algorithm and its particularization to Voronoi diagrams. An example of the results is shown in the next set of figures which has been obtained using convex distance functions. Figure 2.a shows the diagram of two sites with non connected Voronoi regions. Figure 2.b shows the diagram of two sites with a two dimensional bisector. The darkest regions that appear in the image represent the two dimensional parts of the bisector. Figure 2.c shows the diagram obtained using different site shapes and different distances: a point and a segment with the Euclidean metric, a point with a quadrilateral as associated convex shape, and a point with a non-unit circle as associated convex shape.



(a) *Non connected regions.* (b) *Two dimensional bisector.* (c) *Different distances.*

Figure 2: *Convex distance*

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