Convex sets in graphs

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The study of abstract convexity began in the early fifties with the search for an axiom system that defines a convex set and in some way generalises the classical concept of a Euclidean convex set. Numerous contributions to this topic have been made. An extensive survey of this subject can be found in [15].

Among the wide variety of structures that have been studied under abstract convexity are metric spaces, ordered sets or lattices and graphs, the last being the focus of this paper. Several abstract convexities associated with the vertex set of a graph are well-known (see [8]). Their study is of interest in Computational Geometry and has some direct applications to other areas such as, for example, Game Theory (see [4]).

For graph terminology we follow [11]; except that we use vertex instead of point and edge instead of line. All graphs considered here are finite, simple, unweighted and undirected. The *interval* between a pair u, v of vertices in a graph G is the collection of all vertices that lie on some shortest $u-v$ path in G and is denoted by $I_G[u,v]$ or $I[u,v]$ if G is understood. Intervals in graphs have been studied extensively (see [2, 13, 14]) and play an important role in the study of several classes of graphs such as the Ptolemaic graphs or block graphs. A subset S of vertices of a graph is said to be *convex* if it contains the interval between every pair of vertices in S. This definition allows us to study several problems from Euclidean convexity in a finite and discrete setting.

If S is a convex set in a graph, a vertex $p \in S$ is said to be an *extreme point* for S if $S - \{p\}$ is still convex. A vertex in a graph is *simplicial* if its neighbourhood induces a complete subgraph. So p is an extreme vertex for a convex set S if and only if p is simplicial in the subgraph induced by S.

The *convex hull* of a set S of vertices in a graph G is the smallest convex subset of G that contains S and is denoted by $\mathrm{CH}(S)$. It is true, in general, that the convex hull of the extreme points of a vertex set S is contained in S, but equality holds only in special cases. If a graph satisfies this property for every convex subset of the vertex set, it is said to have the *Minkowski-Krein-Milman property.* In [8] it is shown that a graph has this property if and only if it has no induced cycles of length bigger than 3 and has no induced 3-fan (see Figure 1).

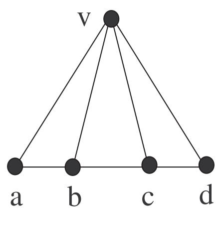

Figure 1: A 3-fan

If a graph G has the Minkowski-Krein-Milman property and S is a convex set of $V(G)$, then we

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can rebuild the set S from its extreme vertices using the convex hull operation. Since this cannot be done with every graph, using only the extreme vertices of a given convex set S, it is natural to ask if it is possible to extend the set of extreme vertices of S to a set that allows us to rebuild S using the vertices in this extended set and the convex hull operation. We answer this question in the affirmative using the collection of 'contour vertices' of a set. To this end, let S be a set of vertices in a graph G and recall that the *eccentricity in* S of a vertex $u \in S$ is given by $ecc_S(u) = \max\{d(u,v) : v \in S\}$ and a vertex $v \in S$ for which $d(u,v) = ecc_S(u)$ is called an *eccentric vertex* for u in S. In case $S = V(G)$, we denote $ecc_S(u)$ by $ecc(u)$. A vertex $u \in S$ is said to be a *contour vertex* of S if $ecc_S(u) \geq ecc_S(v)$ for every neighbour v of u in S. The set of all contour vertices of S is called the *contour* of S and is denoted by $\mathrm{Ct}(S)$. If $S = V(G)$, the set is called the *contour* of G and is denoted by $\mathrm{Ct}(G)$.

The relationship between contour and extreme points is shown in the two following results.

Lemma 1. *Let G be a graph and $S \subseteq V(G)$. Then $\mathrm{Ct}(S)$ contains all extreme vertices of S.*

Proposition 2. *Let G be a distance-hereditary graph without induced 4-cycles. A vertex $x \in V(G)$ is a contour vertex for G if and only if each neighbour v of x which is on a shortest path between x and some eccentric vertex for x satisfies $N(x) \subseteq N(v)$.*

The following result shows that the convex hull of the contour set of a convex set of vertices in a graph is the entire set, without any restriction on the graph. So this result is similar to the Minkowski-Krein-Milman property and holds for all graphs.

Theorem 3. *Let G be a graph and S a convex subset of vertices. Then $S = \mathrm{CH}(\mathrm{Ct}(S))$.*

Now we characterize those graphs that are the contour of some other graph. The following results tells us which graphs are not the contour of any graph.

Proposition 4. *If H is a connected, non-complete graph with radius 1, then H is not the contour of any graph.*

On the other hand, suppose that H is a connected graph with radius greater than 1. We now describe a graph G such that its contour is H, using the construction given in [3]. Let G be the join of H and K_1. Then every vertex of H has eccentricity 2 and the vertex of $G - V(H)$ has eccentricity 1. Hence the vertices of H are precisely the contour vertices of G.

A slightly different construction allows us to obtain a graph with given disconnected contour set such that the eccentricities of the vertices in every component are given numbers at least 2. More precisely, let H be a disconnected graph with components, $H_1, H_2, \ldots, H_k$. Let $n_1, n_2, \ldots n_k$ be k natural numbers such that $n_1 = n_k = \max\{n_1, n_2, \ldots n_k\}$ and $M = \max\{n_1, n_2, \ldots n_k\} \leq 2\min\{n_1, n_2, \ldots, n_k\} = 2m$. Note that these are natural restrictions, because M will be the diameter of the graph G and m will be greater than or equal to the radius. Then there exists a connected graph G such that H is the contour of G and the eccentricity of every vertex in each component H_i of H is equal to n_i. To construct such a graph G we begin with the path $v_1 v_2 \ldots v_{M+1}$ of order $M+1$. Now replace v_1 by H_1 and v_{M+1} by H_k so that all vertices in H_1 are neighbours of v_2 and all vertices in H_k are neighbours of v_M.

Now, for each i, $2 \leq i \leq k-1$ there exists a vertex v_{n_i} on the path such that its eccentricity is $n_i - 1$. We now add H_i to the graph and join all the vertices of H_i to v_{n_i} (see Figure 2). Then $ecc(u_i) = n_i$ for all $u_i \in H_i$, and $\mathrm{Ct}(G) = H$.

In order to find the convex hull of a set S one begins by taking the union of the intervals between pairs of vertices of S, taken over all pairs of vertices in S. We denote this set by $I_G[S]$ or $I[S]$, i.e., $I[S] = \cup_{\{u,v\} \subseteq S} I[u,v]$ and call it the *geodetic closure* of S. One then repeats this procedure with the new set and continues until, for the first time, one reaches a set T for which the geodetic closure is the set itself , i.e., $T = I[T]$. This is then the convex hull of S. If this procedure only has to be performed once, we say that the set S is a *geodetic set* for its convex hull. In general a subset S of a convex set T is a *geodetic set* for T if $I[S] = T$. The notion of a geodetic set for the vertex set of a graph was first defined in [5].

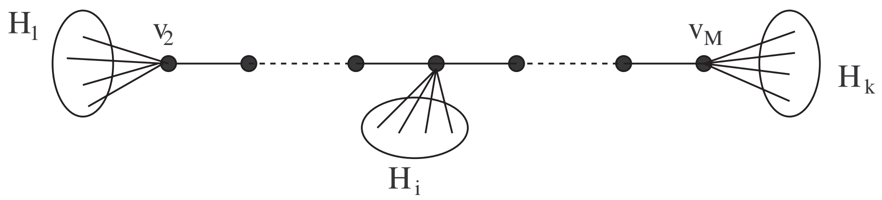

Figure 2: A disconnected contour set

We now focus on geodetic sets in 'distance hereditary graphs'. We first discuss here how these graphs are related to the graphs with the Minkowski-Krein-Milman property. Howorka [12] defined a connected graph G to be *distance hereditary* if for every connected induced subgraph H of G and every two vertices u, v in H, $d_H(u,v) = d_G(u,v)$. In the same paper several characterisations for this class of graphs are given. We state here only one of these which we will use in this paper.

Theorem 5 ([12]). *A connected graph G is distance hereditary if and only if every cycle in G of length at least 5 has a pair of crossing chords.*

Further useful characterizations for this class of graphs were established in [1, 7, 10]. Apart from having elegant characterisations, distance hereditary graphs possess other useful properties. It is a class of graphs for which several NP-hard problems have polynomial solutions. For example, it has been shown in [6, 7] that the Steiner problem for graphs, which is known to be NP-hard (see [9]), can be solved in polynomial time in distance hereditary graphs. Moreover, these graphs are Steiner distance hereditary as was shown in [7]; i.e., the Steiner distance of a set of vertices is the same, in any connected induced subgraph that contains it, as it is in the graph itself.

The class of distance hereditary graphs also properly contains the graphs that possess the Minkowski-Krein-Milman properly since a graph is chordal without an induced 3-fan if and only if it is a distance hereditary graph without an induced 4-cycle. It was shown in [8] that in a chordal graph every non-simplicial vertex lies on a chordless path between two simplicial vertices. If G is a chordless graph without an induced 3-fan, then G is distance hereditary and thus every induced path is necessarily a shortest path. Hence the simplicial vertices for a convex set S in a graph with the Minkowski-Krein-Milman property is a geodetic set for S. We show that the contour vertices of a distance hereditary graph form a geodetic set for the graph.

We need the following Lemma, that relates eccentric and contour points in distance hereditary graphs.

Lemma 6. **(a)** *If G is a distance hereditary graph and $x \in V(G)$, then there is an eccentric vertex for x that is a contour vertex.*
(b) *Let G be a distance hereditary graph without induced 4-cycles. If $x \in V(G)$ is such that $ecc(x) \geq 2$, then each eccentric vertex of x is a contour vertex of G.*

Theorem 7. *Let G be a distance hereditary graph. Then $\mathrm{Ct}(G)$ is a geodetic set for G.*

The graph of Figure 3 shows that Theorem 7 does not hold for graphs in general. Note that the contour set of this graph G is $\mathrm{Ct}(G) = \{v_2, v_5, w\}$ and $v_1 \notin I[\mathrm{Ct}(G)]$.

Indeed if we replace v_1 by a clique of arbitrarily large order and join every vertex in this clique with v_2 and v_8, we see that the ratio $|I[\mathrm{Ct}(G)]|/|V(G)|$ can be made arbitrarily small.

As we mentioned in the introduction, the process of taking geodetic closures starting from a set S of vertices can be repeated to obtain a sequence $S_0, S_1, \ldots$ of sets where $S_0 = S$, $S_1 = I[S]$, $S_2 = I[I[S]]\ldots$. Since $V(G)$ is finite, the process terminates with some smallest r for which $S_r = S_{r+1}$. The set S_r is then the convex hull of S and r is called the *geodetic iteration number*, $gin(S)$, of S. In the graph G of Figure 3, $gin(\mathrm{Ct}(G)) = 2$. It remains an open problem to determine if $gin(\mathrm{Ct}(G))$ can be larger than 2 and indeed if $gin(\mathrm{Ct}(G))$ can be arbitrarily large.

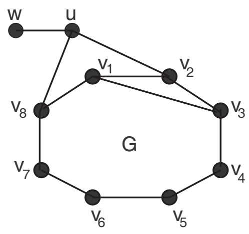

Figure 3: A graph whose contour set is not geodetic

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