The One-Round Voronoi Game Replayed

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Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area Q of aspect ratio $\rho \leq 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of Q closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2+\alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: We show that Barney has a winning strategy for $n \geq 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geq 3$ and $\rho \leq \sqrt{2}/n$, for $n = 2$ and $\rho \leq \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.

Keywords:
Voronoi diagram, Voronoi game, Competitive facility location, NP-hardness.

1 Introduction

When determining success or failure of an enterprise, *location* is one of the most important issues. Probably the most natural way to determine the value of a possible position for a facility is the distance to potential customer sites. Various geometric scenarios have been considered; see the extensive list of references in the paper by Fekete, Mitchell, and Weinbrecht [6] for an overview.

One particularly important issue in location theory is the study of strategies for competing players. See the surveys by Tobin, Friesz, and Miller [8], by Eiselt and Laporte [4], and by Eiselt, Laporte, and Thisse [5].

A simple geometric model for the value of a position is used in the *Voronoi game*, which was proposed by Ahn et al. [1] for the one-dimensional scenario and extended by Cheong et al. [2] to the two- and higher-dimensional case. In this game, a site s "owns" the part of the playing arena that is closer to s than to any other site. Both considered a two-player version with a finite arena Q. The players, White ("Wilma") and Black ("Barney"), place points in Q; Wilma plays first. No point that has been occupied can be changed or reused by either player. Let W be the set of points that were played by the end of the game by Wilma, while B is the set of points played by Barney. At the end of the game, a Voronoi diagram of $W \cup B$ is constructed; each player wins the total area of all cells belonging to points in his or her set. The player with the larger total area wins.

Ahn et al. [1] showed that for a one-dimensional arena, i.e., a line segment $[0, 2n]$, Barney can win the *n-round game*, in which each player places a single point in each turn; however, Wilma can keep Barney's winning margin arbitrarily small. This differs from the *one-round game*, in which both players get a single turn with n points each: Here, Wilma can force a win by playing the odd integer points $\{1, 3, \ldots, 2n-1\}$; again, the losing player can make the margin as small as he wishes. The used strategy focuses on "key points". The question raised in the end of that paper is whether a similar notion can be extended to the two-dimensional scenario. We will see in Section 3 that in a certain sense, this is indeed the case.

Cheong et al. [2] showed that the two- or higher-dimensional scenario differs significantly: For sufficiently large $n \geq n_0$ and $\rho = 1$, the second player has a winning strategy that guarantees at least a fixed fraction of $1/2+\alpha$ of the total area. Their proof used a clever combination of probabilistic arguments to show that Barney will do well by playing

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a random point. The paper gives rise to some interesting open questions:

- How large does n_0 have to be to guarantee a winning strategy for Barney? Wilma wins for $n = 1$, but it is not clear whether there is a single n_0 for which the game changes from Wilma to Barney, or whether there are multiple changing points.
- Barney wins for sufficiently "fat" arenas, while Wilma wins for the degenerate case of a line. How exactly does the outcome of the game depend on the aspect ratio of the playing board?
- What happens if the number of points played by Wilma and Barney are not identical?
- What configurations of white points limit the possible gain of black points? As candidates, square or hexagonal grids were named.
- What happens for the multiple-round version of the game?
- What happens for asymmetric playing boards?

For rectangular boards and arbitrary values of n, we will show when Barney can win the game. If the board Q has aspect ratio ρ with $\rho \leq 1$, we prove the following:

- Barney has a winning strategy for $n \geq 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geq 3$ and $\rho \leq \sqrt{2}/n$, for $n = 2$ and $\rho \leq \sqrt{3}/2$, and for $n = 1$.

- If Wilma does not play her points on an orthogonal grid, then Barney wins the game.

In addition, we hint at the difficulties of more complex playing boards by showing the following:

- If Q is a polygon with holes, and Wilma has made her move, it is NP-hard to find a position of black points that maximizes the area that Barney wins.

This result is also related to recent work by Dehne, Klein, and Seidel [3] of a different type: They studied the problem of placing a single black point within the convex hull of a set of white points, such that the resulting black Voronoi cell in the unbounded Euclidean plane is maximized. They showed that there is a unique local maximum.

The rest of this paper is organized as follows. After some technical preliminaries in Section 2, Section 3 shows that Barney always wins if Wilma does not place her points on a regular orthogonal grid. This is used in Section 4 to establish our results on the critical aspect ratios. Section 5 presents some results on the computational complexity of playing optimally in a more complex board. Some concluding thoughts are presented in Section 6.

For this 4-page abstract, all proofs have been omitted; a full version of the paper [7] is available electronically.

2 Preliminaries

In the following, Q is the playing board. Q is a rectangle of *aspect ratio* ρ, which is the ratio of the length of the smaller side divided by the length of the longer side. Unless noted otherwise (in some parts of Section 5), both players play n points; W denotes the n points played by Wilma, while B is the set of n points played by Barney. All distances are measured according to the Euclidean norm. For a set of points P, we denote by $V(P)$ the (Euclidean) Voronoi diagram of P. We call a Voronoi diagram $V(P)$ a regular grid if

- all Voronoi cells are rectangular, congruent and have the same orientation;

- each point $p \in P$ lies in the center of its Voronoi cell.

If e is a Voronoi edge, $C(e)$ denotes a Voronoi cell adjacent to e. If $p \in P$, then $C(p)$ denotes the Voronoi cell of p in $V(P)$. $\partial C(p)$ is the boundary of $C(p)$ and $|C(p)|$ denotes the area of $C(p)$. $|e|$ denotes the length of an edge e. Let x_p and y_p denote the x- and y-coordinates of a point p.

3 A Reduction to Grids

As a first important step, we reduce the possible configurations that Wilma may play without losing the game. The following holds for boards of any shape:

Lemma 1 *If $V(W)$ contains a cell that is not point symmetric, then Barney wins.*

The following theorem is based on this observation and will be used as a key tool for simplifying our discussion in Section 4.

Theorem 2 *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

4 Critical Aspect Ratios

In this section we prove the main result of this paper: if $n \geq 3$ and $\rho > \sqrt{2}/n$, or $n = 2$ and $\rho > \sqrt{3}/2$, then Barney wins. In all other cases, Wilma wins. The proof proceeds by a series of lemmas. We start by noting the following easy observation.

Lemma 3 *Barney wins, if and only if he can place a point p that steals an area strictly larger than* $|Q|/2n$ *from W.*

Next we take care of the case $n = 2$; this lemma will also be useful for larger n, as it allows further reduction of the possible arrangements Wilma can choose without losing.

Lemma 4 *If* $n = 2$ *and* $\rho > \sqrt{3}/2$*, then Barney wins. If the aspect ratio is smaller, Barney loses.*

The gain for Barney is small if ρ is close to $\sqrt{3}/2$. Computer experiments have been used to compute the gain for Barney for values of $\rho > \sqrt{3}/2$. Not surprisingly, the largest gain was found for $\rho = 1$. If the board has size 1 by 1, Barney can gain an area of approximately 0.2548 with his first point, by placing it at (0.66825,0.616). as illustrated in Figure 1(a).

Lemma 5 *Suppose that the board is rectangular and that* $n = 4$*. If Wilma places her point on a regular* 2×2 *grid, Barney can gain* 50.78% *of the board.*

The value in the above lemma is not tight. For example, if Wilma places her point in a 2 by 2 grid on a square board, we can compute the area that Barney can gain with his first point. If Barney places it at (0.5,0.296), he gains approximately 0.136. For an illustration, see Figure 1(b). By placing his remaining three points at $(0.25 - 4\varepsilon/3, 0.25)$, $(0.25 - 4\varepsilon/3, 0.75)$, and $(0.75 + 4\varepsilon/3, 0.75)$ Barney can gain a total area of size of around $0.511 - \varepsilon$ for arbitrary small positive ε. For non-square boards, we have found larger wins for Black. This suggests that Barney can always gain more than 51% of the board if Wilma places her four points in a 2 by 2 grid.

The above discussion has an important implication:

Corollary 6 *If* $n \geq 3$*, then Wilma can only win by placing her points in a* $1 \times n$ *grid.*

This sets the stage for the final lemma:

Lemma 7 *Let* $n \geq 3$*. Barney can win if* $\rho > \sqrt{2}/n$*; otherwise, he loses.*

Computational experiments have confirmed that Barney wins the largest area with his first point if he places it at $(0, (4r - 2\sqrt{r^2 + 6})/3)$.

Theorem 8 *If* $n \geq 3$ *and* $\rho > \sqrt{2}/n$*, or* $n = 2$ *and* $\rho > \sqrt{3}/2$*, then Barney wins. In all other cases, Wilma wins.*

5 A Complexity Result

The previous section resolves most of the questions for the one-round Voronoi game on a rectangular board. Clearly, there are various other questions related to more complex boards; this is one of the questions raised in [2]. Lemma 1 still applies if Wilma's concern is only to avoid a loss. Moreover, it is clear that all of Wilma's Voronoi cells must have the same area. For many boards, both of these conditions may be impossible to fulfill. It is therefore natural to modify the game by shifting the critical margin that decides a win or a loss. We show in the following that it is NP-hard to decide whether Barney can beat a given margin for a polygon with holes, and all of Wilma's stones have already been placed. (In a non-convex polygon, possibly with holes, we measure distances according to the geodesic Euclidean metric, i.e., along a shortest path within the polygon.)

Theorem 9 *For a polygon with holes, it is NP-hard to maximize the area Barney can claim, even if all of Wilma's points have been placed.*

6 Conclusion

We have resolved a number of open problems dealing with the one-round Voronoi game. There are still several issues that remain open. What can be said about achieving a fixed margin of win in all of the cases where Barney can win? We believe that our above techniques can be used to resolve this issue. As we can already quantify this margin if Wilma plays a grid, what is still needed is a refined version of Lemma 1 and Theorem 2 that guarantees a fixed margin as a function of the amount that Wilma deviates from a grid. Eventually, the guaranteed margin should be a function of the aspect ratio. Along similar lines, we believe that it is possible to resolve the question stated by [2] on the scenario where the number of points played is not equal.

Probably the most tantalizing problems deal with the multiple-round game. Given that finding an optimal set of points for a single player is NP-hard,

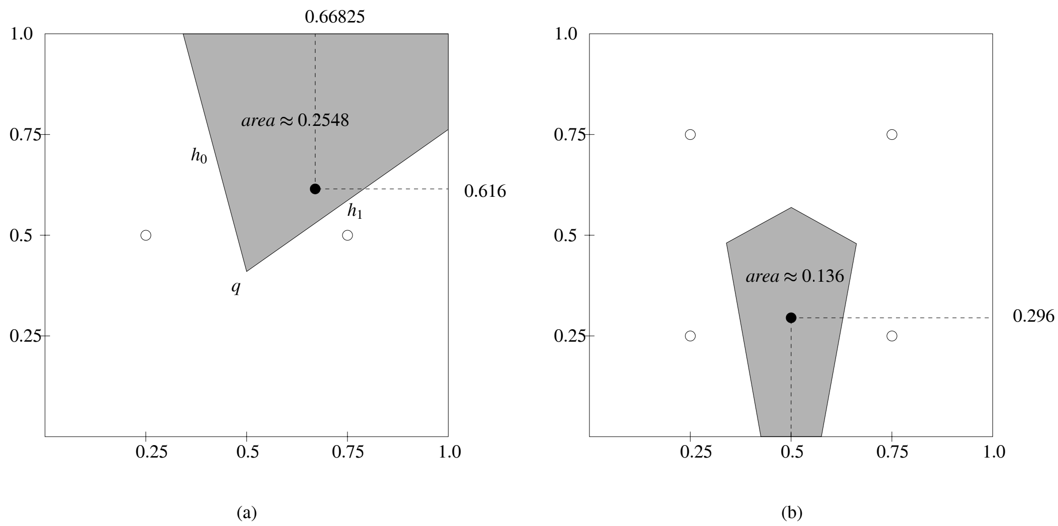

Figure 1: Barney has gained more than a quarter (a) more than an eighth (b) of the playing surface.

it is natural to conjecture that the two-player, multiple round game is PSPACE-hard. Clearly, there is some similarity to the game of Go on an $n \times n$ board, which is known to be PSPACE-hard [9] and even EXPTIME-complete [10] for certain rules.

However, some of this difficulty results from the possibility of capturing stones. It is conceivable that at least for relative simple (i.e., rectangular) boards, there are less involved winning strategies. Our results from Section 4 show that for the cases where Wilma has a winning strategy, Barney cannot prevent this by any probabilistic or greedy approach: Unless he blocks one of Wilma's key points by placing a stone there himself (which has probability zero for random strategies, and will not happen for simple greedy strategies), she can simply play those points like in the one-round game and claim a win. Thus, analyzing these key points may indeed be the key to understanding the game.

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