Graphs of triangulations and perfect matchings

M. Houle[1] F. Hurtado[2] M. Noy[2] E. Rivera-Campo[3]

Abstract

Given a set P of points in the plane, the graph of triangulations $\mathcal{T}(P)$ has a vertex for every triangulation of P, and two of them are adjacent if they differ by a single edge exchange. In this paper we prove that the subgraph $\mathcal{T}_{\mathcal{M}}(P)$ of $\mathcal{T}(P)$, consisting of all triangulations of P that admit a perfect matching, is connected. A main tool in our proof is a result of independent interest, namely that the graph $\mathcal{M}(P)$ that has as vertices the non-crossing perfect matchings of P and two of them are adjacent if their symmetric difference is a single non-crossing cycle, is also connected.

Keywords. Triangulation. Perfect matching. Non-crossing graph.

1 Introduction

Given a set P of points in the plane, the graph of triangulations $\mathcal{T}(P)$ has a vertex for every triangulation of P, and two of them are adjacent if they differ by a single edge exchange. Graphs of triangulations have been widely studied; see for example [5, 6]. In particular, it is well-known that $\mathcal{T}(P)$ is a connected graph.

In this paper we study the subgraph $\mathcal{T}_{\mathcal{M}}(P)$ of $\mathcal{T}(P)$, consisting of all triangulations of P that admit a perfect matching. Not every triangulation contains a perfect matching, so in general $\mathcal{T}_{\mathcal{M}}(P)$ is a proper subgraph of $\mathcal{T}(P)$. Our main result is that the graph $\mathcal{T}_{\mathcal{M}}(P)$ is connected for any set P in general position. In other words, we show that any two triangulations of P containing a perfect matching can be connected through a sequence of edge exchanges, always resulting in triangulations containing a perfect matching.

In order to prove our main result, we first prove another result of independent interest, which we now describe. Given a set P in the plane of even cardinality, a perfect matching in P is said to be non-crossing if no two of its edges intersect. The graph $\mathcal{M}(P)$ has as vertices the non-crossing perfect matchings of P, and two of them are adjacent if their symmetric difference is a single non-crossing cycle. The case were P is in convex position was studied in [4]. We show that the graph $\mathcal{M}(P)$ is connected for any set P in general position; this is the key ingredient for proving that $\mathcal{T}_{\mathcal{M}}(P)$ is a connected graph.

The rest of the paper is organized as follows. Section 2 contains the results on graphs of perfect matchings, and Section 3 on graphs of triangulations containing perfect matchings. Our graph theory terminology follows that of [2]. Throughout the paper we assume that all point sets are in general position, that is, no three points are collinear.

[1]IBM Research, Tokyo Research Laboratory, Japan, meh@trl.ibm.com.

[2]Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Spain, hurtado@ma2.upc.es, noy@ma2.upc.es. Partially supported by Projects DGES-SEUID PB98-0933, MCYT-BFM2001-2340, MCYT-FEDER-BFM2002-0557 and Gen. Cat 2001SGR00224.

[3]Departamento de Matemáticas. Universidad Autónoma Metropolitana-I, México, erc@xanum.uam.mx. Part of the research was done while this author was on sabbatical leave visiting the Universitat Politècnica de Catalunya with grants by MECD-Spain and CONACYT-México.

2 Graphs of perfect matchings

Let P be a set of $2m$ points in general position in the plane. The symmetric difference of two non-crossing perfect matchings in P is a set of alternating cycles; some of these cycles may have crossings, see Figure 1. We say that two perfect matchings M_1 and M_2 differ in a *single alternating non-crossing cycle exchange* if their symmetric difference is a single non-crossing cycle; for brevity we say that M_2 is obtained from M_1 by performing a *flip*.

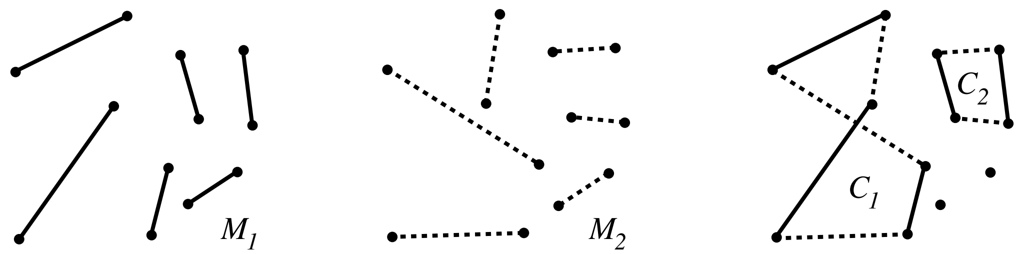

Figure 1: Two matchings M_1 and M_2; their symmetric difference (right) is the union of two alternating cycles C_1 and C_2, but only C_2 is non-crossing.

The graph of non-crossing perfect matchings $\mathcal{M}(P)$ of P is the graph with one vertex for each non-crossing perfect matching of P, in which two matchings are adjacent if and only if one can be obtained from the other by a flip. The requirement that the cycle involved in the exchange is non-crossing is not only a natural one, but it is critical when applying Theorem 2.1 in the next section.

Theorem 2.1. *For any set P of $2m$ points in general position in the plane, the graph $\mathcal{M}(P)$ is a connected graph.*

3 Graphs of triangulations

Let P be a set of points in the plane in general position. The graph of triangulations $\mathcal{T}(P)$ is the graph with one vertex for each triangulation of P, in which two triangulations T_1 and T_2 are adjacent if and only there are edges $e \in T_1 \setminus T_2$ and $f \in T_2 \setminus T_1$ such that $T_2 = T_1 \setminus \{e\} \cup \{f\}$. In other words, T_2 is obtained from T_1 by replacing the diagonal of a convex quadrilateral by the other diagonal.

For a non-crossing set E of line segments with endpoints in P, let $\mathcal{T}(P, E)$ be the subgraph of $\mathcal{T}(P)$ induced by the set of triangulations of P that contain all edges in E.

Lemma 3.1. *Let P be a set of points in general position in the plane, E be a non-crossing set of line segments with ends in P and $e \notin E$ be a line segment, also with ends in P, and such that $E \cup \{e\}$ is a non-crossing set. For each triangulation T of P that contains all edges in E there is a triangulation S of P containing $E \cup \{e\}$ which is connected to T in $\mathcal{T}(P, E)$.*

Theorem 3.2. *$\mathcal{T}(P, E)$ is a connected graph for any set P of points in general position in the plane and any non-crossing set E of line segments with ends in P.*

For a set P of $2m$ points in general position in the plane, let $\mathcal{T}_{\mathcal{M}}(P)$ be the subgraph of $\mathcal{T}(P)$, induced by the set of triangulations of P that admit a perfect matching. Notice that a set P may admit some triangulations which contain a matching while some others do not contain any (Figure 2).

Theorem 3.3. *$\mathcal{T}_{\mathcal{M}}(P)$ is a connected graph for any set P of $2m$ points in general position in the plane.*

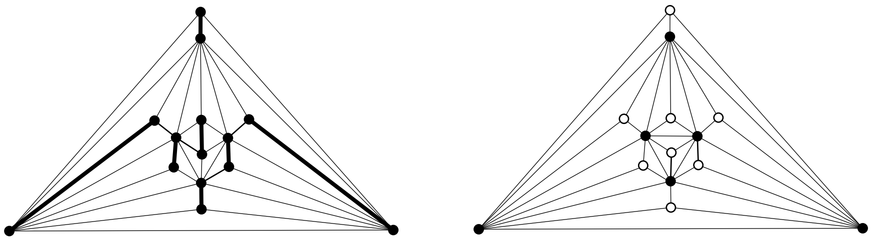

Figure 2: The triangulation on the left part of the figure contains a perfect matching (solid lines), but the triangulation on the right part does not contain any, because the 8 independent white nodes are adjacent only to the 6 black nodes.

4 Conclusions

Our definition of adjacency of the graph of non-crossing matchings $\mathcal{M}(\mathcal{P})$ of P via a single alternating non-crossing cycle exchange contains no constraint on the length of the cycle. Nevertheless, as pointed out in [3], for the purposes of optimization, enumeration, and random generation, it is desirable that the transformation making a class connected is as local as possible, which somehow amounts to use an exchange of constant size at each step. Therefore it is natural to consider a graph of matchings $\mathcal{M}'(P)$ in which only exchanges in cycles of length $\ell = 4$ (say) are considered. It is an open problem to decide whether such graph is connected for some constant value of ℓ. For $\ell = 4$ we have been able to prove that the corresponding graph contains no isolated point; yet even this modest fact required quite a long proof.

Finally, there other subgraphs of $\mathcal{T}(P)$ for which it would be interesting to know whether they induced a connected subgraph or not. For instance, the set of 3-connected triangulations of P (see [1] for a related problem), or the set of triangulations with minimum degree at least k.

References

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