Orthogonal Segment Stabbing*

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1 Introduction

We consider various stabbing problems for orthogonal line segments in the plane. Let H, V be sets of horizontal and vertical segments. In the first problem that we consider, one has to find a subset S of $H \cup V$ of minimal size, such that for each segment $s \in H \cup V$, either $s \in S$ or s is stabbed by some segment s' in S. We refer to this problem as the Orthogonal Segment Dominating Set problem (OSDS). This problem is closely related to a guarding problem posed by Frank Hoffmann. In Hoffmann's problem, the sets H and V represent a connected network of horizontal and vertical streets, and one has to find a patrol path of minimal length, such that every street point (i.e., a point on an input segment) is seen from some point on the path. In another problem that we consider, one has to find a minimal subset of *vertical* segments that stab all horizontal segments (assuming such a set exists). We refer to this problem as the Orthogonal Segment Vertex Cover problem (OSVC).

In Section 2 we show that OSDS and OSVC are NP-complete, by reducing them to vertex dominating set for planar bipartite graphs and to vertex cover for planar graphs, respectively. Both reductions are based on a representation theorem (Theorem 2.1) due to Ben-Arroyo Hartman et al. [1] and to de Fraysseix et al. [4], and on the corresponding algorithmic result of de Fraysseix et al. [5]. This theorem states that any planar bipartite graph $G = (A, B; E)$ can be represented by a collection of $|A|$ horizontal segments and $|B|$ vertical segments, such that a horizontal segment and a vertical segment corresponding to vertices a and b stab each other if and only if $(a, b) \in E$.

We obtain two variants of OSVC by extending the segments in V to downwards-directed rays, or by extending the segments in H to rightwards-directed rays. In other words, let H be a set of horizontal segments and let R be a set of downwards-directed rays. In the *stabbing segments with rays problem* one has to find a minimal subset of R that stabs all segments in H, and in the *stabbing rays with segments problem* one has to find a minimal subset of H that stabs all rays in R. We present polynomial-time solutions to both these problems that are based on dynamic programming. In this extended abstract we only sketch the solution to the former problem (Section 3), and completely omit the solution to the latter problem, which is significantly more complicated.

Both these problems have an equivalent formulation involving a set T of (possibly intersecting) axis-parallel rectangles and a set P of points. Given such sets, the corresponding piercing problem (i.e., compute a minimal subset P' of P, such that each rectangle is pierced by a point in P') and

*Research by M.K. and J.M. is partially supported by grant no. 2000160 from the U.S.-Israel Binational Science Foundation. Research by M.K. and Y.N. is also partially supported by the MAGNET program of the Israel Ministry of Industry and Trade (LSRT consortium). Research by J.M. is also partially supported by HRL Laboratories, NASA Ames Research, National Science Foundation, Northrop-Grumman, Sandia National Labs, Sun Microsystems.

the corresponding covering problem (i.e., compute a minimal subset T' of T, such that each point is covered by a rectangle in T') are known to be NP-complete (see [3]).

We restrict these problems by adding the assumptions that all rectangles in T are "hanging" from a mutual line. With this assumption, the piercing problem becomes the stabbing segments with rays problem, and the covering problem becomes the stabbing rays with segments problem. (For each point $p \in P$ draw a downwards-directed ray emanating from p, and for each rectangle $t \in T$ keep its bottom edge only).

2 Stabbing Segments with Segments

In this section we prove that OSVC and OSDS are NP-complete. Our proofs are based on the following representation theorem due to Ben-Arroyo Hartman et al. [1] and to de Fraysseix et al. [4], and on the corresponding algorithmic result of de Fraysseix et al. [5].

Theorem 2.1 *[1, 4] Any planar bipartite graph $G = (A, B; E)$ can be represented by a set $I(G)$ of $|A|$ disjoint horizontal segments and $|B|$ disjoint vertical segments, such that two segments a and b stab each other if and only if $(a, b) \in E$. The set $I(G)$ is said to be a* grid representation *of G.*

Theorem 2.2 *[5] Let $G = (A, B; E)$ be a planar bipartite graph. One can compute a grid representation $I(G)$ of G in $O(|A| \cup |B|)$ time.*

2.1 OSVC is NP-complete

Let H be a set of horizontal segments and let V be a set of vertical segments. The Orthogonal Segment Vertex Cover problem (OSVC) asks for a minimal subset of V that stabs all segments in H.

Theorem 2.3 *OSVC is NP-Complete.*

Proof: We prove that OSVC is NP-complete by reducing it to Minimum Vertex Cover for planar graphs, which is known to be NP-complete [7]. Let $G = (V, E)$ be a planar graph. By placing a new vertex b_e in the middle of each edge e of E, we obtain a planar bipartite graph $G' = (A, B; F)$, where $A = V$, B corresponds to E, and there is an arc between a and b if and only if a is adjacent to the edge of G corresponding to b. It follows from the construction that G' is planar and bipartite.

Next we compute a grid representation $I(G') = V \cup H$ of G', using the linear-time algorithm of [5]. (The existence of such a representation follows from Theorem 2.1 above.) This completes the reduction, since a minimum vertex cover for G becomes a minimum subset of A that dominates all vertices in B, which in turn becomes a solution to OSVC (i.e., a minimum subset of V that stabs all segments in H). □

2.2 OSDS is NP-complete

Let H be a set of horizontal segments and let V be a set of vertical segments. The Orthogonal Segment Dominating Set problem (OSDS) asks for a minimal subset of $S = H \cup V$, such that for each segment $s \in H \cup V$, either $s \in S$ or s is stabbed by some segment s' in S.

Theorem 2.4 *OSDS is NP-Complete.*

Proof: We prove that OSDS is NP-complete by reducing it to Minimum Dominating Set for planar bipartite graphs. Let $G = (A, B; E)$ be a planar bipartite graph and let $I(G) = V \cup H$ be a grid representation of G. (By Theorem 2.1 and Theorem 2.2, $I(G)$ exists and can be computed in linear time.) Assuming that Minimum Dominating Set for planar bipartite graphs is NP-complete we are done, since a minimum dominating set for G becomes a solution to $OSDS$. However, our search for a proof of the NP-completeness of Minimum Dominating Set for planar bipartite graphs was unsuccessful, so we include below a simple proof of this claim. This proof appears implicitly in a paper by Kariv and Hakimi [2]; it is based on the NP-completeness of Minimum Vertex Cover for planar graphs [7].

We cut each edge (u, v) of the planar graph G and insert a square a,x,b,y as shown in Figure 1. Clearly, the graph G' that is obtained remains planar. It is also bipartite. Moreover, a minimum vertex cover for G can easily be transformed to a minimum dominating set for G', by adding, for each edge (u, v) of G, either the vertex a or the vertex b. If u is not in the minimum vertex cover for G, we add a, if v is not in the minimum vertex cover, we add b, and otherwise we pick one of the two arbitrarily. □

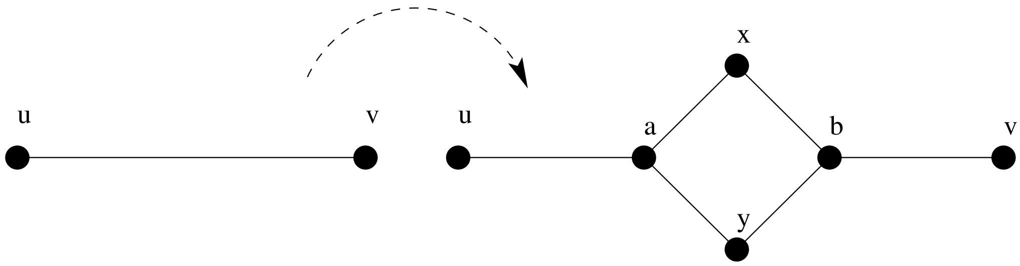

Figure 1: Construction for proof of Theorem 2.4.

3 Stabbing Segments with Rays

Let S be a set of horizontal segments and let R be a set of downwards-directed rays. For a ray $r \in R$, let $x(r)$ and $y(r)$ denote the x and y values, respectively, of r's origin. For a segment $s \in S$ let $h(s)$ denote its height, and let $l(s)$ and $r(s)$ denote the x values of its left and right endpoints, respectively. A ray is *maximal* if it is above the highest segment.

In the Stabbing Segments with Rays problem (SSR) one has to find a minimal subset $R' \subseteq R$, such that, for each segment $s \in S$, there exists a ray $r \in R'$ that stabs it.

Lemma 3.1 *Any solution consists of a maximal ray and solutions for two disjoint subproblems.*

Proof: Let R^* be a solution for $< S, R >$ and let $r \in R^*$ be a ray that stabs a segment of maximum height in S. (r is by definition a maximal ray.) Any segment that is not stabbed by r is either completely to the left or completely to the right of r, and any ray in $R \setminus \{r\}$ is either to the left or to the right of r. Therefore if R_l, R_r are solutions for the resulting left and right subproblems, respectively, then $\{r\} \cup R_l \cup R_r$ is a solution for $< S, R >$. □

We present a top-down recursive algorithm for computing a solution for SSR. By employing dynamic programming a bottom-up non-recursive algorithm can be obtained, that is more efficient in terms of space.

A subproblem $< Left, Height, Right >$ contains all segments and rays of the problem $< S, R >$ that are contained in the interior of the corresponding (infinite) vertical slab, and whose height is less or equal to $Height$. According to some simple observations that are omitted from this version, we may assume that the problem $< S, R >$ is given by $< 0, |S| + |R|, 2|S| + |R| + 1 >$, i.e., all endpoints are unique integers in the range $[1, 2|S| + |R|]$ for the x-values, and in the range $[1, |S| + |R|]$ for the y-values.

The algorithm examines the highest ray r within the subproblem $< Left, Height, Right >$. If r is not maximal (within the subproblem), then the subproblem has no solution. Else, the algorithm compares between the sizes of the subsets of R obtained by either including r or not including r (see Figure 2). If r is included, then the solution obtained is the union of $\{r\}$ and the solutions for the subproblems $< Left, Height - 1, x(r) >$ and $< x(r), Height - 1, Right >$ (that do not contain segments that are stabbed by r). And if r is not included, then the solution for the subproblem is $< Left, Height - 1, Right >$. We omit the analysis of the algorithm from this version, and only mention that its time complexity is $O(|S|^3)$.

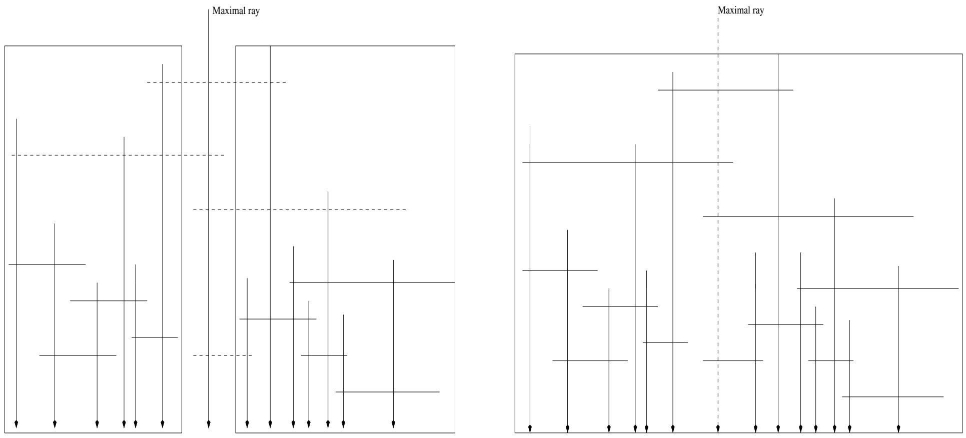

Figure 2: Subproblems resulting from (a) choosing the maximal ray r and (b) not choosing r.

Acknowledgement

We wish to thank Arie Tamir for helpful discussions.

References

[1] I. Ben-Arroyo Hartman, I. Newman and R. Ziv, *On grid intersection graphs*, Discrete Mathematics 87, 1991, 41–52.

[2] O. Kariv and L. Hakimi, *Algorithmic approach to network location problems*, SIAM J. Applied Mathematics 37, 1979, 513–538.

[3] R. J. Fowler, M. S. Paterson and, S. L. Tanimoto, *Optimal packing and covering in the plane are NP-complete*, Information Processing Letters, 12(3), 1981, 133–137.

[4] H. de Fraysseix, P. O. de Mendez, and J. Pach *Representation of planar graphs by segments* Intuitive Geometry, Coll. Math. Soc. J. Bolyai 63, 1991, 109–117.

[5] H. de Fraysseix, P. O. de Mendez, and J. Pach *A left-first search algorithm for planar graphs* Discrete Computational Geometry, 13, 1995, 459–468.

[6] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman and co., New York.

[7] M. R. Garey and D. S. Johnson, *Some simplified NP-complete graph problems*, Theoretical Comp. Sci., 1, 1976, 237–267.

[8] R. Hassin and N. Megiddo, *Approximation algorithms for hitting objects by straight lines*, Discrete Applied Mathematics, 30, 1991, 29–42.