Affine representations of abstract convex geometries

Kenji Kashiwabara* Masataka Nakamura† Yoshio Okamoto‡

Abstract

A convex geometry is a combinatorial abstract model introduced by Edelman and Jamison which captures a combinatorial essence of "convexity" shared by some structures including finite point sets, partially ordered sets, trees, rooted graphs. In this paper, we introduce a generalized convex shelling, and we show that any convex geometry can be represented as a generalized convex shelling. This is "the representation theorem for convex geometries" similar to "the representation theorem for oriented matroids" by Folkman and Lawrence. An important feature is that our representation theorem is affine-geometric while that for oriented matroids is topological. Namely our representation theorem indicates the intrinsic simplicity of convex geometries.

1 Introduction

Some abstract models of geometric concepts are known to be useful. For example, a matroid is considered as the abstraction of linear dependence and plays important roles in finite geometry, coding theory, combinatorial optimization and so on [11]. Another example is an oriented matroid, which is considered as the abstraction of affine (and linear) dependence and which captures essences of convex polytopes, point configurations, hyperplane arrangements and so on [1]. Oriented matroids play an important role in theory of convex polytopes, discrete geometry, computational geometry and so on, and they are known to be quite powerful models.

One of the most important theorems in oriented matroid theory is the "topological representation theorem" by Folkman and Lawrence [7]. The topological representation theorem states that: any simple oriented matroid can be represented as a "pseudohyperplane arrangement." So, in principle, when we investigate an oriented matroid, we only have to look at the corresponding pseudohyperplane arrangement. A recent study by Swartz [12] revealed the topological representation of matroids, saying that every simple matroid can be represented as the arrangement of homotopy spheres.

In this paper, we will study yet another example of combinatorial abstraction of geometric concepts, namely a convex geometry. A convex geometry was introduced by Edelman and Jamison [6] as an abstraction of convexity, and it can be seen as a "dual" (or a "polar" or a "complement") of an antimatroid [4]. A convex geometry has been appearing in papers not only on discrete geometry or combinatorics but also on social choice theory ([10] for example) or mathematical psychology ([5] for a detailed treatment). Also, convex geometries form a greedily solvable special case of a certain optimization problem [2].

In this paper, we will show a representation theorem for convex geometries. The theorem says that any convex geometry can be represented as a "generalized convex shelling." Since a generalized convex shelling is defined in a purely affine-geometric manner, this theorem gives an affine-geometric representation of a convex geometry. Since neither an affine-geometric representation theorem for matroids nor for oriented matroids is known, our affine-

*Department of Systems Science, Graduate School of Arts and Sciences, The University of Tokyo, 3–8–1, Komaba, Meguro, Tokyo, 153–8902, Japan. E-mail: kashiwa@graco.c.u-tokyo.ac.jp.

†Department of Systems Science, Graduate School of Arts and Sciences, The University of Tokyo, 3–8–1, Komaba, Meguro, Tokyo, 153–8902, Japan. E-mail: nakamura@klee.c.u-tokyo.ac.jp.

‡Institute of Theoretical Computer Science, Department of Computer Science, ETH Zürich, ETH Zentrum, CH-8092, Zürich, Switzerland. E-mail: okamotoy@inf.ethz.ch. Supported by the Berlin-Zürich Joint Graduate Program "Combinatorics, Geometry, and Computation" (CGC), financed by ETH Zürich and the German Science Foundation (DFG).

geometric representation theorem for convex geometries indicates the intrinsic simplicity of convex geometries. As well as the topological representation theorem for oriented matroids plays a significant role in theory of oriented matroids, our theorem will play a similar role in theory of convex geometries.

2 Convex geometries and the representation theorem

In this section, we will give a definition of a convex geometry, which was introduced by Edelman and Jamison [6], and will state our theorem precisely.

Let E be a nonempty finite set. A family $\mathcal{L}$ of subsets of E is called a *convex geometry* on E if $\mathcal{L}$ satisfies the following three axioms:

(L1) $\emptyset \in \mathcal{L}$ and $E \in \mathcal{L}$;

(L2) if $X, Y \in \mathcal{L}$, then $X \cap Y \in \mathcal{L}$;

(L3) if $X \in \mathcal{L} \setminus \{E\}$ then there exists some $e \in E \setminus X$ such that $X \cup \{e\} \in \mathcal{L}$.

Two convex geometries $\mathcal{L}_1$ on E_1 and $\mathcal{L}_2$ on E_2 are *isomorphic* if there exists a bijection $\psi : E_1 \to E_2$ such that $\psi(X) \in \mathcal{L}_2$ if and only if $X \in \mathcal{L}_1$.

Let us look at some examples of convex geometries.

Example 2.1 (convex shelling). Let Q be a finite set of points in $\mathbb{R}^d$, and define

$$\mathcal{L} = \{X \subseteq Q : \mathrm{conv}(X) \cap Q = X\}.$$

Then, we can see that $\mathcal{L}$ is a convex geometry on Q, and we say this kind of convex geometries is a *convex shelling*. A convex geometry isomorphic to some convex shelling on a finite point set Q is also called a convex shelling.

Example 2.2 (poset shelling). Let E be a partially ordered set endowed with a partial order $\preceq$, and define $\mathcal{L} = \{X \subseteq E : e \in X$ and $f \preceq e$ imply $f \in X\}$. Then we can see that $\mathcal{L}$ is a convex geometry on E, and we say this kind of convex geometries is a *poset shelling*.

Example 2.3 (tree shelling). Let V be the vertex set of a (graph-theoretic) tree T, and define $\mathcal{L} = \{X \subseteq V : u, v \in X \Rightarrow$ a unique path connecting u and v only uses vertices in $X\}$. Then we can see that $\mathcal{L}$ is a convex geometry on V, and we say this kind of convex geometries is a *tree shelling*.

Example 2.4 (graph search). Let $G = (V, E)$ be a rooted connected graph with root $r \in V$, and define $\mathcal{L} = \{X \subseteq V \setminus \{r\} : v \in V \setminus X \Rightarrow$ v can be reached from r by a path only using vertices in $V \setminus X\}$. Then we can see that $\mathcal{L}$ is a convex geometry on $V \setminus \{r\}$, and we say this kind of convex geometries is a *graph search*.

For other various examples of convex geometries, see [6] or [9].

Here, we will give yet another example of convex geometries, which was not given explicitly before.

Example 2.5 (generalized convex shelling). Let P and Q be finite point sets in $\mathbb{R}^d$. Assume that $\mathrm{conv}(P) \cap Q = \emptyset$ and particularly that $P \cap Q = \emptyset$. Then define

$$\mathcal{L} = \{X \subseteq Q : \mathrm{conv}(X \cup P) \cap Q = X\}.$$

We say $\mathcal{L}$ is the *generalized convex shelling on Q with respect to P*. If $P = \emptyset$, this just gives a convex shelling. So, as the name indicates, a generalized convex shelling is a generalization of a convex shelling. While at first sight it is not so obvious that a generalized convex shelling is indeed a convex geometry, that can be shown. (Here we omit the proof.)

Our main theorem will be as follows. This says that the class of convex geometries coincides with the class of generalized convex shellings, although convex geometries arise from diverse objects as we saw.

Theorem 2.1. *Any convex geometry is isomorphic to some generalized convex shelling.*

The main concern of this paper is the proof of Theorem 2.1. In the next section, for the proof of Theorem 2.1, we will construct finite sets P_0 and Q_0 of points from a given convex geometry $\mathcal{L}$ so that $\mathcal{L}$ can be isomorphic to the generalized convex shelling on Q_0 with respect to P_0.

3 Construction of point sets

For our construction, we will use rooted circuits of a convex geometry. So at the beginning of this section, we will introduce rooted circuits.

A rooted circuit of a convex geometry was originally introduced by Korte and Lovász [8].

In order to define a rooted circuit, we need some more technical words. For a convex geometry $\mathcal{L}$ on E and $A \subseteq E$, the *trace* of $\mathcal{L}$ on A is defined as $\mathrm{Tr}(\mathcal{L}, A) = \{X \cap A : X \in \mathcal{L}\}$. A *rooted set* is a pair (X, r) of a set X and an element r of X. A *rooted subset* of E is a rooted set (X, r) such that $X \subseteq E$.

Here comes the definition of a rooted circuit. Let $\mathcal{L}$ be a convex geometry on E. A rooted subset (C, r) of E is called a *rooted circuit* of $\mathcal{L}$ if $\mathrm{Tr}(\mathcal{L}, C) = 2^C \setminus \{C \setminus \{r\}\}$. We denote the family of rooted circuits of a convex geometry $\mathcal{L}$ by $\mathcal{C}(\mathcal{L})$.

Now we are ready for our construction. We will construct point sets P_0 and Q_0 from a given convex geometry $\mathcal{L}$ on E so that $\mathcal{L}$ can be isomorphic to the generalized convex shelling on Q_0 with respect to P_0.

Let us say that $|E| = n$. We will take the $(n-1)$-dimensional space $\mathbb{R}^{n-1}$. For each element $e \in E$, we take a point $\boldsymbol{q}(e) \in \mathbb{R}^{n-1}$ such that the points $\boldsymbol{q}(e) \in \mathbb{R}^{n-1}$ $(e \in E)$ can be affinely independent. Namely, it should hold that for any $\{\mu_e \in \mathbb{R} : e \in E\}$ with $\sum_{e \in E} \mu_e = 0$,

$$\sum_{e \in E} \mu_e \boldsymbol{q}(e) = \mathbf{0} \;\Rightarrow\; \mu_e = 0 \text{ for all } e \in E.$$

(So $\mathrm{conv}(\{\boldsymbol{q}(e) : e \in E\})$ is an $(n-1)$-dimensional simplex.) Also for each rooted circuit $(C, r) \in \mathcal{C}(\mathcal{L})$ of $\mathcal{L}$ we put a point $\boldsymbol{p}(C, r) \in \mathbb{R}^{n-1}$ determined as

$$\boldsymbol{p}(C, r) = |C|\boldsymbol{q}(r) - \sum_{e \in C \setminus \{r\}} \boldsymbol{q}(e). \qquad (1)$$

Note that $\boldsymbol{q}(r)$ lies in the relative interior of $\mathrm{conv}(\{\boldsymbol{q}(e) : e \in C \setminus \{r\}\} \cup \{\boldsymbol{p}(C, r)\})$ for any rooted circuit $(C, r) \in \mathcal{C}(\mathcal{L})$. In this way, we have set up $|E| + |\mathcal{C}(\mathcal{L})|$ points in $\mathbb{R}^{n-1}$.

Let $P_0 = \{\boldsymbol{p}(C, r) : (C, r) \in \mathcal{C}(\mathcal{L})\}$ and $Q_0 = \{\boldsymbol{q}(e) : e \in E\}$. Then $P_0 \cap Q_0 = \emptyset$. Now our claim is as follows.

Claim 3.1. *For P_0 and Q_0 constructed above, the generalized convex shelling on Q_0 with respect to P_0 is isomorphic to $\mathcal{L}$.*

This claim proves Theorem 2.1.

To illustrate the construction, we will look at examples for $n = 3$. For $n = 3$ we have six non-isomorphic convex geometries. Let $E = \{1, 2, 3\}$ for example. Below we enumerate all of the six non-isomorphic convex geometries on $\{1, 2, 3\}$ together with their rooted circuits. $\mathcal{L}_1 = 2^{\{1,2,3\}}$ and $\mathcal{C}(\mathcal{L}_1) = \emptyset$; $\mathcal{L}_2 = \mathcal{L}_1 \setminus \{\{1, 3\}\}$ and $\mathcal{C}(\mathcal{L}_2) = \{(\{1, 2, 3\}, 2)\}$; $\mathcal{L}_3 = \mathcal{L}_2 \setminus \{\{3\}\}$ and $\mathcal{C}(\mathcal{L}_3) = \{(\{2, 3\}, 2)\}$; $\mathcal{L}_4 = \mathcal{L}_3 \setminus \{\{2, 3\}\}$ and $\mathcal{C}(\mathcal{L}_4) = \{(\{1, 3\}, 1), (\{2, 3\}, 2)\}$; $\mathcal{L}_5 = \mathcal{L}_3 \setminus \{\{1\}\}$ and $\mathcal{C}(\mathcal{L}_5) = \{(\{1, 2\}, 2), (\{2, 3\}, 2)\}$; $\mathcal{L}_6 = \mathcal{L}_4 \setminus \{\{2\}\}$ and $\mathcal{C}(\mathcal{L}_6) = \{(\{1, 2\}, 1), (\{1, 3\}, 1), (\{2, 3\}, 2)\}$.

Figure 1 depicts the construction of the point sets for these examples.

4 Idea of the proof

Because of the limitation of the pages, we will just describe an idea of the proof of Claim 3.1. The entire proof will appear in the full-paper version. In this section, any proof will be omitted.

Let $\mathcal{L}'$ be the generalized convex shelling on Q_0 with respect to P_0. The first thing that we should care about is that the constructed point sets P_0 and Q_0 actually satisfy the precondition of generalized convex shellings, namely $\mathrm{conv}(P_0) \cap Q_0 = \emptyset$. In fact, we can show that this is the case.

Next, we want to establish a bijection ψ from E to Q_0 such that ψ can be an isomorphism between $\mathcal{L}$ and $\mathcal{L}'$. As it is natural, we will set $\psi(e) = \boldsymbol{q}(e)$ for each $e \in E$. We want to show that ψ is an expected isomorphism between $\mathcal{L}$ and $\mathcal{L}'$.

To show that, we will use a result by Dietrich [3, 4] which is a characterization of a convex geometry in terms of the family of rooted circuits. Therefore, in order to show that ψ is an isomorphism, we will show that ψ maps a rooted circuit of $\mathcal{L}$ to a rooted circuit of $\mathcal{L}'$ bijectively. From the characterization by Dietrich [3, 4], we can find that it suffices to show the following two lemmas for our purpose.

Lemma 4.1. 1. *In the setting above, for any rooted circuit $(C, r) \in \mathcal{C}(\mathcal{L})$, there exists $(C', \boldsymbol{r}') \in \mathcal{C}(\mathcal{L}')$ such that $C' \subseteq \psi(C)$ and $\boldsymbol{r}' = \psi(r)$.*

2. *In the setting above, for any rooted circuit $(C', \boldsymbol{r}') \in \mathcal{C}(\mathcal{L}')$, there exists $(C, r) \in \mathcal{C}(\mathcal{L})$ such that $C \subseteq \psi^{-1}(C')$ and $r = \psi^{-1}(\boldsymbol{r}')$.*

We need more facts to prove Lemma 4.1. Actually, for a proof of Lemma 4.1.2 we use the concept of a closure operator which appears

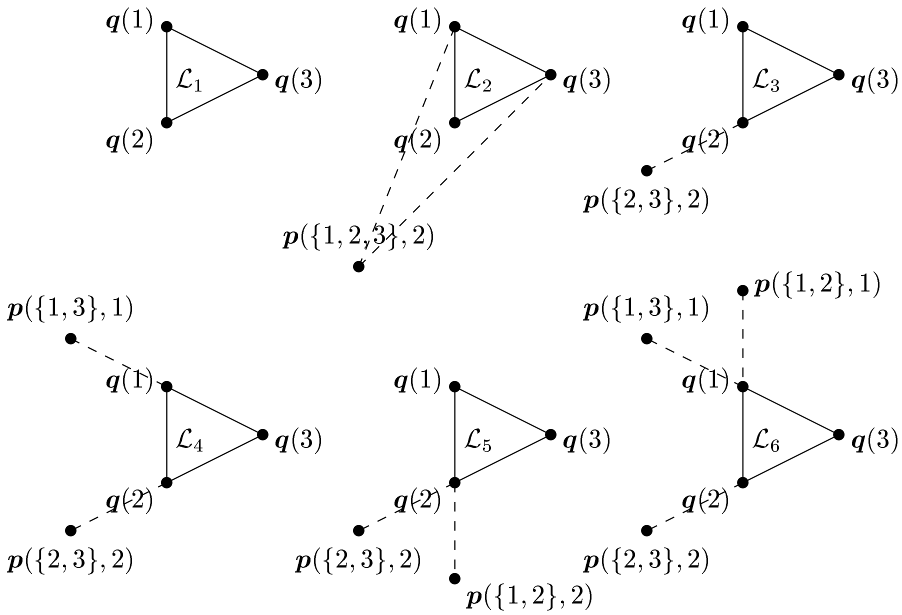

Figure 1: Construction of point sets for $n = 3$.

in the theory of convex geometries (or closure spaces more generally).

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