The Voronoi diagram for n disjoint spherical sites of $O(1)$ different radii has complexity $O(n^2)$ (extended abstract)

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Abstract

It is proved that if S is a set of n disjoint spherical sites in $\mathbf{R}^3$, of at most k different radii, then the cell owned by a smallest site has fewer than $3^{k-1}n$ different faces.

It follows that the Voronoi diagram of S has complexity $O(n^2)$, assuming a bound on the number of different radii among the sites in S.

It is also shown that without the bound on the number of different radii, the cell owned by a point site can have complexity $\Omega(n^2)$.

1 Voronoi diagrams

This paper considers the Voronoi diagrams of spherical sites in $\mathbf{R}^3$. For a general survey of Voronoi diagrams see, e.g., [1]. The current state of knowledge about the complexity of Voronoi diagrams, in 3 dimensions, is scanty. It is known to be $O(n^2)$ for n point sites, and this bound is tight. When the sites are straight lines, the complexity is known to be $o(n^{2+\epsilon})$ for all $\epsilon > 0$, granted that either the distance function is polyhedral, based on a fixed convex polyhedron [2], or the distance is Euclidean but the lines are in $O(1)$ different directions [3].

The so-called *sites* will be a set S of n disjoint closed balls in $\mathbf{R}^3$. The *Voronoi diagram* of S is the set of points in $\mathbf{R}^3$ which have more than one site closest to them. The Voronoi diagram is a 2-dimensional complex with faces, edges, and vertices. The faces are connected subsets of what we call *bisectors*.

1.1 Definition. Let B and B' be disjoint spherical sites (point sites are allowed), Then the (B, B')*-bisector* is the Voronoi diagram of $\{B, B'\}$, that is, the set of points equidistant from B and B'.

1.2 Lemma. *If B and B' are spherical sites with radii r, r' respectively, where $r \geq r' \geq 0$, then the bisector of B and B' is a plane if $r = r'$ and a (single sheet of a 2-sheeted) hyperboloid of revolution, whose axis is the line joining their centres, if $r > r'$.*

In either case, the bisector partitions $\mathbf{R}^3$ into two regions, and that containing B' is convex. See Figure 1. □

1.3 Corollary. *Suppose that S is a set of disjoint spherical sites whose minimum radius is r_1. Let S' be the set of spherical sites obtained by replacing every site B in s by a site with same centre and radius $r - r_1$, where r is the radius of B.*

Then $\mathrm{Vor}(S) = \mathrm{Vor}(S')$, *and the smallest sites in S' are point sites.* □

1.4 Definition. Let B be one of the sites in a set S of disjoint spherical sites. The *Voronoi cell* of B consist of all points which are as close, or closer to, B than to any other site in S.

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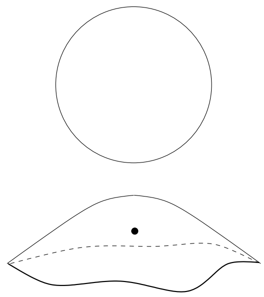

Figure 1: The bisector is a hyperboloid of revolution.

Clearly, the Voronoi diagram is the union of boundaries of cells of all sites in S.
The complexity of the Voronoi diagram for point sites is known:

1.5 Proposition. *If S is a set of n point sites, then* $\mathrm{Vor}(S)$ *has n cells and $O(n^2)$ faces, edges, and vertices. This bound is tight.* □

1.6 Lemma. *Let $\mathcal{C}$ be a collection of sets S of disjoint spherical sites in $\mathbf{R}^3$. Let $M(n)$ denote the maximum complexity of* $\mathrm{Vor}(S)$ *for all $S \in \mathcal{C}$ such that $|S| = n$. Then $M(n)$ is $O(n^2)$ if and only if for every $S \in \mathcal{C}$,* $\mathrm{Vor}(S)$ *contains a cell of complexity $O(|S|)$.* □

2 Re-inflating deflated sites

It is our aim to show that when S is a set of disjoint spherical sites with at most k distinct radii, and B is a site of minimum radius in S, then its cell in $\mathrm{Vor}(S)$ has at most $3^{k-1}n$ faces. With k fixed it can be regarded as having $O(n)$ different faces, and hence its complexity is $O(n)$. This is enough (Lemma 1.6) to ensure that $\mathrm{Vor}(S)$ has complexity $O(n^2)$, when the number of different radii occurring among the sites in S is bounded.

(2.1) Inflating sites. We imagine the sites being 'inflated' to their correct size: an increasing parameter r is given, and $S(r)$ is the set of sites with their radius bounded by r. As r increases, the sites inflate until all have reached their correct radius. We study how the Voronoi diagram evolves.

2.2 Definition. Let S be a set of n spherical sites with centres c_i and radii r_i. For any $r \geq 0$, the *r-bounded version* $S(r)$ of S is the set of n sites whose centres are c_i but whose radii are $\min(r_i, r)$.

We can assume (Corollary 1.3) that S contains a point site. For the remainder of this section, p will denote a point site in S.

2.3 Definition. Suppose that B is a site in $S(r)$. If the corresponding site in S has radius $> r$ then we say B is *expanding*, otherwise it is *stable*.

A face, edge, or vertex of $\mathrm{Vor}(S(r))$ is called *stable, transient static, or moving* according as all sites closest to it are stable, all are expanding, or some but not all are expanding, respectively.

2.4 Definition. $C(r)$ will denote the cell owned by p in $\mathrm{Vor}(S(r))$.

2.5 Lemma. $C(r)$ *is convex. (Immediate from Lemma 1.2.)* □

We consider the evolution of $C(r)$ as r increases (up to the maximum radius occurring in S).

$C(0)$ is a convex polyhedron with at most $n-1$ faces. As r increases, some of these faces become curved, and new faces appear and disappear.

2.6 Lemma. *The only way a new face can be introduced to $C(r)$ is when a bisector passes through a stable or transient static vertex of $C(r)$.*

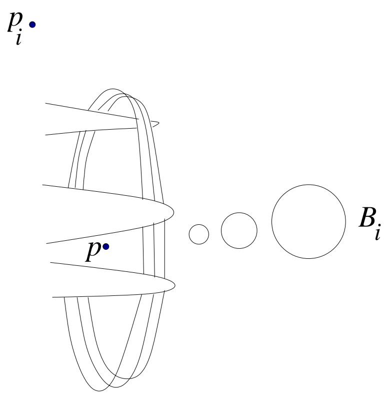

Figure 2: Cell of p has $\Omega(n^2)$ incident edges.

Sketch proof. We must consider all possible ways in which the number of faces of $C(r)$ can change. We classify them as follows:

(A) Several cases which turn out to be impossible:

- (Ai) A face gets separated when two opposite edges touch.
- (Aii) A face gets introduced when two bisectors touch.
- (Aiii) A face gets introduced when a bisector touches an edge, and the face begins to separate the edge.

(B) A bisector passes through a moving vertex.

(C) A bisector passes through a transient static vertex.

(D) A bisector passes through a stable vertex.

In case (B) a face disappears, and cases (C) and (D) are as predicted. □

2.7 Corollary. *$C(r)$ has at most 3^{k-1} faces.*

Sketch proof. Let

$$0 = r_1 < r_2 < \ldots < r_k$$

be the different radii occurring among the sites in S. Whenever $C(r)$ acquires a new face, a vertex was lost under case (C) or (D). Suppose $r_s \leq r < r_{s+1}$. If case (D) applied, then that vertex existed in $C(r_s)$, and we can assume by induction that there are at most $2 * 3^s$ vertices in $C(r_s)$. If the vertex did not exist in $C(r_s)$, then the vertex must have been introduced at some time r', $r_s < r' < r$, in an event of type (B). But then a face was lost from $C(r')$, which can be offset against the gain of a new face by $C(r)$ through a type (C) event. □

3 The bound on number of radii is essential

Let p be a point site located at $(0, 0, 0)$. Let H be the unit sphere centred at $(0, 0, 1)$. H touches p. Place n balls B_j centred on the x-axis at $(1/2^j, 0, 0)$ and tangent to H: they are disjoint. Place n point sites p_i around the circle $x = 0, y^2 + z^2 = 4$. The points p, p_i and the balls B_j form a set S of $2n + 1$ sites.

The circle $E: \ x = 0, y^2 + z^2 = 1$ is a degenerate edge where the cell of p in $\text{Vor}(S)$ meets those of the p_i and the B_j.

Slightly expand the sites B_j, and displace the point sites p_i slightly towards p. The effect is to replace E by $n - 1$ circles close to E, and the point sites p_i split these $n - 1$ circles into n edges each. The cell of p has more than $n(n - 1)$ incident edges. The idea is illustrated in Figure 2.

4 References

1. Franz Aurenhammer (1990). Voronoi diagrams— A survey of a fundamental geometric data structure. *ACM computing surveys* **23.3**, 345–405.

2. L. Paul Chew, Klara Kedem, Micha Sharir, Boris Tagansky, and Emo Welzl (1998). Voronoi diagrams of lines in 3-space under polyhedral convex distance functions. *Journal of Algorithms 29*, 238–255.

3. Vladlen Koltun and Micha Sharir (2002). Three dimensional Euclidean Voronoi diagrams of lines with a fixed number of orientations. *Proc. 18th European Workshop on Computational Geometry*, 1–3.