# The polytope of non-crossing graphs on a planar point set *

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## 1 Introduction

The set of (straight-line, or geometric) non-crossing graphs with a given set of vertices $\mathcal{A}$ in the plane is of interest in Computational Geometry, Geometric Combinatorics, and related areas. In particular, much effort has been directed towards enumeration, counting and optimization on the set of maximal such graphs, that is to say, triangulations of $\mathcal{A}$. But little is known about the poset structure of the set of all non-crossing subgraphs under inclusion. In this paper we associate to $\mathcal{A}$ a polytope whose face poset contains the poset of non-crossing graphs on $\mathcal{A}$ embedded in a very nice way.

The construction is based on [5], where a polyhedron $\overline{X_f}(\mathcal{A})$ of dimension $2n-3$ with face poset (opposite to) that of *pointed* non-crossing graphs on $\mathcal{A}$ is constructed. A straight-line graph embedded in the plane is called pointed if the edges incident to every vertex span an angle smaller than 180 degrees. Let $n$ denote the size of $\mathcal{A}$, $n_b$ and $n_i$ the number of points in the boundary and the interior of its convex hull, respectively. The polyhedron $\overline{X_f}(\mathcal{A})$ has a unique maximal bounded face $X_f(\mathcal{A})$, of dimension $2n_i + n_b - 3$, there called the *polytope of pointed pseudo-triangulations* of $\mathcal{A}$.

Our main new ingredient is that we consider "marked" non-crossing graphs, meaning non-crossing graphs together with the specification of a subset of their pointed vertices. With similar ideas but with $n$ extra coordinates for the $n$ possible marks, we get a polyhedron $\overline{Y_f}(\mathcal{A})$ of dimension $3n-3$ with a unique maximal bounded face $Y_f(\mathcal{A})$ of dimension $3n_i + n_b - 3$. The face $F$ in the statement of Theorem 2.5 is precisely the polytope $X_f(\mathcal{A})$, which arises by setting to 0 the $n$ new coordinates, corresponding to marks.

The technical tools both in our construction and in [5] are *pseudo-triangulations* of planar point sets and their relation to structural rigidity of non-crossing graphs. Pseudo-

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triangulations, first introduced by Pocchiola and Vegter around 1995 (see [3]), have by now been used in many Computational Geometry applications, among them visibility [4, 3], ray shooting [1], and kinetic data structures [2]. Streinu [6] introduced the *minimum* or *pointed pseudo-triangulations*, and used them to prove the Carpenter's Rule Theorem. Pointed pseudo-triangulations turn out to coincide with the maximal non-crossing and pointed graphs; that is to say, with the vertices of the polytope $X_f(\mathcal{A})$. Our method extends that construction and $Y_f(\mathcal{A})$ is the *polytope of pseudo-triangulations*, since its vertices correspond to all pseudo-triangulations of $\mathcal{A}$.

## 2 Overview of the method and results

In the sequel we assume our point set $\mathcal{A}$ to be in general position, although the results are proved in the full paper for point sets in degenerate position as well. Let $n_i$ and $n_b$ be the number of points of $\mathcal{A}$ in the interior and boundary of $\text{conv}(\mathcal{A})$, respectively, and let $n = n_i + n_b$. We define:

**Definition 2.1 (Flips in pseudo-triangulations)** Let $T$ be a pseudo-triangulation of $\mathcal{A}$. We call *flips* in $T$ the following three types of operations, all producing pseudo-triangulations.

- (*Deletion flip*). The removal of an edge $e \in T$, if $T \setminus e$ is a pseudo-triangulation.
- (*Insertion flip*). The insertion of an edge $e \notin T$, if $T \cup e$ is a pseudo-triangulation.
- (*Diagonal flip*). The exchange of an edge $e \in T$, if $T \setminus e$ is not a pseudo-triangulation, for the unique edge $e'$ such that $(T \setminus e) \cup e'$ is a pseudo-triangulation.

The *graph of pseudo-triangulations* of $\mathcal{A}$ has as vertices all the pseudo-triangulations of $\mathcal{A}$ and as edges all flips of any of the types.

**Proposition 2.2** *The graph of pseudo-triangulations of $\mathcal{A}$ is connected and regular of degree $3n_i + n_b - 3 = 3n - 2n_b - 3$.*

As happened with pointed pseudo-triangulations, Proposition 2.2 suggests that the graph of pseudo-triangulations of $\mathcal{A}$ may be the skeleton of a simple polytope of dimension $3n_i + n_b - 3$. As a step towards this result we look at what the face poset of such a polytope should be. The polytope being simple means that we want to regard each pseudo-triangulation $T$ as the upper bound element in a Boolean poset of order $3n - 3 - 2n_b$. This number equals the number of interior edges plus interior pointed vertices in $T$:

**Definition 2.3** A *marked graph* on $\mathcal{A}$ is a geometric graph with vertex set $\mathcal{A}$ together with a subset of its vertices, that we call "marked". We call a marked graph *non-crossing* if it is non-crossing as a graph and marks arise only in pointed vertices.

We call a non-crossing marked graph *fully-marked* if it is marked at all pointed vertices. If, in addition, it is a pseudo-triangulation, then we call it a *fully-marked pseudo-triangulation*, abbreviated as *f.m.p.t.*

We start defining a linear cone $\overline{Y_0}(\mathcal{A})$ by one inequality for each possible edge and each point of $\mathcal{A}$. Its $\binom{n+1}{2}$ facets are then translated using the entries of a vector $f$ in $\mathbb{R}^{\binom{n+1}{2}}$ to produce a polyhedron $\overline{Y_f}(\mathcal{A})$ which has as unique maximal bounded face a polytope $Y_f(\mathcal{A})$. Our proof goes by analyzing the necessary and sufficient conditions for $f$ to produce a polytope with the desired properties. We get the next result, where flips in fully-marked pseudo-triangulations are defined in the natural way from those in pseudo-triangulations:

**Theorem 2.4 (The polyhedron of marked non-crossing graphs)** *If $f$ is a valid choice of parameters, then there is a simple polyhedron $\overline{Y_f}$ of dimension $3n-3$ whose face poset equals (the opposite of) the poset of non-crossing marked graphs on $\mathcal{A}$. In particular:*

(a) *Vertices of the polyhedron are in 1-to-1 correspondence with fully-marked pseudo-triangulations of $\mathcal{A}$.*

(b) *Bounded edges correspond to flips of interior edges or marks in fully-marked pseudo-triangulations, i.e., to fully-marked pseudo-triangulations with one interior edge or mark removed.*

(c) *Extreme rays correspond to fully-marked pseudo-triangulations with one convex hull edge or mark removed.*

We prove valid choices of $f$ to be the interior of a convex polyhedron defined by $\binom{n}{4}$ strict inequalities and give an explicit choice. Then, from the existence of a valid $f$ and the Theorem above, the following result is concluded:

**Theorem 2.5 (The polytope of all pseudo-triangulations)** *Let $Y_f(\mathcal{A})$ be the face of $\overline{Y_f}(\mathcal{A})$ defined turning into equalities the equations which correspond to convex hull edges or convex hull points of $\mathcal{A}$, and assume $f$ to be a valid choice. Then:*

1. *$Y_f(\mathcal{A})$ is a simple polytope of dimension $3n-2n_b-3$ whose 1-skeleton is the graph of pseudo-triangulations of $\mathcal{A}$. (In particular, $Y_f(\mathcal{A})$ is the unique maximal bounded face of $\overline{Y_f}(\mathcal{A})$).*
2. *Let $F$ be the face of $Y_f(\mathcal{A})$ defined by turning into equalities the equations corresponding to interior points. Then, the complement of the star of $F$ in the proper face-poset of $Y_f(\mathcal{A})$ equals the poset of non-crossing graphs on $\mathcal{A}$ which use all the convex hull edges.*

This result, which proves the claims advanced in the introduction, deserves some words of explanation:

- Since convex hull edges are irrelevant to crossingness, the poset of *all* non-crossing graphs on $\mathcal{A}$ is the direct product of the poset in the statement and a Boolean poset of rank $n_b$.
- The equality of posets in Theorem 2.5.2 reverses inclusions. Maximal non-crossing graphs (triangulations of $\mathcal{A}$) correspond to minimal faces (vertices of $Y_f(\mathcal{A})$).

- By "proper" face poset of a polytope we mean that the polytope itself is not considered a face. We remind the reader that the *star* of a face $F$ is the union of all the facets (maximal proper faces) containing $F$.

- We give a fully explicit facet description of $Y_f(\mathcal{A})$. It lives in $\mathbb{R}^{3n}$ and is defined by 3 linear equalities and $\binom{n}{2} + n$ linear inequalities in which those $2n_b$ corresponding to convex hull edges and vertices of $\mathcal{A}$ have to be turned into equalities, thus providing an affine subspace of dimension $3n - 3 - 2n_b$, as stated. The face $F$ is the one obtained turning into equalities also the $n_i$ equations corresponding to interior points.

It is worth mentioning that our results have some rigid-theoretic consequences. Namely:

**Theorem 2.6** *Let $T$ be a pseudo-triangulation of a planar point set $A$. Let $G$ be its underlying graph. Then:*

1. *$G$ is infinitesimally rigid, hence rigid and generically rigid.*
2. *There are at least $2k + 3l$ edges of $T$ incident to any subset of $k$ pointed plus $l$ non-pointed vertices of $T$.*

If the pseudo-triangulation is pointed, then it has $2n - 3$ edges and parts (2) is just the Laman characterization of isostatic graphs in the plane as graphs with $2n - 3$ edges with every $k$ vertices incident to at least $2k$ edges. In particular, Theorem 2.6 generalizes Ileana Streinu's result [6] that pointed pseudo-triangulations are isostatic graphs.

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