Tiling Polyominoes with Squares that Touch the Boundary

Andreas Spillner

Institut für Informatik, Friedrich-Schiller-Universität Jena, 07743 Jena

spillner@minet.uni-jena.de

Abstract

We study the problem of tiling a polyomino P with squares such that every tile S has a nonempty intersection with the boundary of P. We are especially interested in tilings with a minimum number of squares and in efficient algorithms to find such tilings.

1 Introduction

Our setting in the present paper will be the plane $\mathbb{R}^2$. For a subset $M \subseteq \mathbb{R}^2$ we denote by $int(M)$ the interior of M, by $bd(M)$ the boundary of M and by $conv(M)$ the convex hull of M. We will consider subsets of $\mathbb{R}^2$ which are usually called polyominoes [3].

Definition 1.1 The set $\mathfrak{C}$ of cells in the plane is defined as

$$\mathfrak{C} = \{conv(\{(k,l),(k+1,l),(k,l+1),(k+1,l+1)\}) : k,l \in \mathbb{Z}\}$$

Definition 1.2 $P \subseteq \mathbb{R}^2$ is called polyomino if and only if there exists a finite nonempty set $\mathfrak{B} \subseteq \mathfrak{C}$ such that $P = \cup_{C \in \mathfrak{B}} C$ and $int(P)$ is connected.

Definition 1.3 A rectangle $R \subseteq \mathbb{R}^2$ is called a cell rectangle if and only if R is a polyomino. A cell rectangle which is a square is called a cell square.

Definition 1.4 Let P be a polyomino. A set $\mathfrak{T}$ of cell squares is called a boundary touching tiling or BTT of P if and only if

1. $P = \cup_{S \in \mathfrak{T}} S$
2. $\forall S_1, S_2 \in \mathfrak{T}\ (S_1 \neq S_2 \ \Rightarrow\ int(S_1) \cap int(S_2) = \emptyset)$
3. $\forall S \in \mathfrak{T}\ (S \cap bd(P) \neq \emptyset)$

The tiling of bounded subsets $M \subseteq \mathbb{R}^2$ with squares that touch the boundary of M was to our knowledge first studied in [4]. However the definition of a BTT given there allows tilings with squares of arbitrary side length and tilings with an infinite number of squares. Then it is shown, that every open set $M \subseteq \mathbb{R}^2$ such that M is a homeomorph of an open disk or a homeomorph of an open disk with one hole admits a (possibly infinite) tiling with squares that touch the boundary of M. On the other hand there exists an open set $M \subseteq \mathbb{R}^2$ such that M is a homeomorph of an open disk with two holes which can not be tiled with squares that touch the boundary of M.

We will restrict ourselves to simple polyominoes, which are homeomorphs of a closed disk, and we use the definition of a BTT given above. With the same arguments as used in [4] one can show, that every simple polyomino admits a BTT. Now by defintion a BTT is finite and we can ask for BTT-s with a minimum number of tiles for a given simple polyomino.

Definition 1.5 Let P be a simple polyomino. $\mu(P) = min(\{|\mathfrak{T}| : \mathfrak{T} \text{ is a BTT of } P\})$

In the literature one finds results concerning the tiling of cell rectangles with a finite number of squares under various conditions. So for example: the number of squares has to be as small as possible [2], the squares of the tiling have to be pairwise incongruent [1]. The latter tilings are called perfect.

2 Some properties of boundary touching tilings

First we note, that for a given polyomino there need not be a unique BTT. Consider for example the rectangle with side lengths 2 and 8 displayed in figure 1. The integers inside the squares denote their side lengths.

Figure 1: Different BTT-s for a given polyomino.

Definition 2.1 Let P be a simple polyomino. A straight line segment C with endpoints x and y is called a cut through P if and only if $C \cap bd(P) = \{x, y\}$ and C dissects P into two simple polyominoes P_1 and P_2, that is $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = C$.

Lemma 2.1 Let P be a simple polyomino and $\mathfrak{T}$ a BTT of P with $|\mathfrak{T}| > 1$. Then there exists a cut C through P such that $C \subseteq \cup_{S\in\mathfrak{T}} bd(S)$.

This lemma can be proved by contradiction. We suppose there is no such cut C. Then we can construct a simple closed curve H such that $H \subseteq int(P)$ and $H \subseteq \cup_{S\in\mathfrak{T}} bd(S)$. But this contradicts the fact that every $S \in \mathfrak{T}$ touches the boundary of P.

Corollary 2.1 Let R be a cell rectangle and $\mathfrak{T}$ a BTT of R with $|\mathfrak{T}| > 1$. Then there exist at least two distinct squares $S_1, S_2 \in \mathfrak{T}$ such that the side length of S_1 equals the side length of S_2 and is minimal among the side lengths of the squares in $\mathfrak{T}$.

Corollary 2.2 For a cell rectangle a BTT with more than one tile can never be a perfect tiling.

In the next section we want to use the cut property of a BTT to give an algorithm, which for some subclass of polyominoes admits a rather efficient computation of BTT-s with the mimimum number of tiles. Such BTT-s we will call minimum BTT-s for short.

3 Boundary touching tilings with the minimum number of tiles

Our idea for an algorithm to compute a minimum BTT for a given simple polyomino P is rather straight forward. We try every possible cut through P. From a cut C we get the two subpolyominoes $P_1(C)$ and $P_2(C)$. We compute a minimum BTT $\mathfrak{T}_1(C)$ for $P_1(C)$ and a minimum BTT $\mathfrak{T}_2(C)$ for $P_2(C)$. Then $\mathfrak{T}(C) = \mathfrak{T}_1(C) \cup \mathfrak{T}_2(C)$ is a BTT for P. To obtain a minimum BTT for P we only have to search for a cut C through P such that $|\mathfrak{T}(C)|$ is minimum.

However if the minimum BTT-s for the subpolyominoes are computed by a recursive call to the same algorithm we will end up with a very slow algorithm. Unfortunately in general we were not able to overcome this difficulty. Only for a very restricted subclass of simple polyominoes we could do better.

Definition 3.1 A polyomino P is called orthogonally convex if and only if for every horizontal or vertical straight line S the set $S \cap P$ is empty or a straight line segment.

Now when we have an orthogonally convex polyomino P consisting of n cells, the number of possible subpolyominoes resulting from cuts is bounded by a polynomial in n. Therefore we can apply the dynamic programming technique and obtain an algorithm the running time of which is bounded by a polynomial in n too.

4 Rectangles

In this section we present some results concerning cell rectangles, because on the one hand, as has been already mentioned in the introduction, there has been done quiet a lot of research on tiling rectangles by squares. On the other hand we think these results are interesting in themselves and leave to us some open questions.

First cell rectangles are special orthogonally convex polyominoes. So we can apply the algorithm given above to compute minimum BTT-s. But we can tune the algorithm so that we end up with a running time in $O(n^2)$ where n is the number of cells the rectangle consists of.

Definition 4.1 Let a and b be positive integers such that $a \leq b$. We set $\mu(a,b) = \mu(R)$ where R is a cell rectangle with side lengths a and b.

The function $\mu(\cdot,\cdot)$ shows some kind of periodicity:

Lemma 4.1 Let a and b be positive integers such that $a\lfloor \frac{a}{2} \rfloor + 2a \leq b$. Then $\mu(a,b) = \mu(a,b-a)+1$.

Remark 4.1 The equation in the above lemma does not hold in general. For example we have $\mu(17,19) = 11$ but $\mu(17,36) = 10$.

Another interesting problem is to give bounds for the function $\mu(\cdot,\cdot)$. So far we were only successful in giving a lower bound. We employ a technique which has been already used in [1] and [2]. We are given a rectangle R with side lengths a and b such that $a \leq b$ and a and b are relatively prime. For any BTT $\mathfrak{T}$ of R with $|\mathfrak{T}| = m$ we construct a graph $G(\mathfrak{T})$ with m edges. It is known that the number of spanning trees of $G(\mathfrak{T})$ is bounded from below by b. On the other hand we were able to show that the number of spanning trees of $G(\mathfrak{T})$ is at most the mth Fibonacci number F_m. Hence we have $b \leq F_m$ or with other words a logarithmic lower bound on m in terms of b.

5 Concluding remarks

There remain some unsettled questions: Does an efficient algorithm exist, which given a simple polyomino P computes a minimum BTT for P? What about an upper bound on the function $\mu(\cdot,\cdot)$? The work done in [2] might indicate, that there are connections to number theory. Finally we remark that the condition that every square in the tiling has to touch the boundary is really a restriction with respect to the number of squares in a minimum tiling. Consider for example the cell rectangle R with side lengths 11 and 13. A minimum BTT of R contains 8 squares, but we can tile R with 6 cell squares. Can we say something about how many more squares we need due to the condition of boundary touching?

References

[1] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte. The dissection of rectangles into squares. *Duke Mathematical Journal*, 7:312–340, 1940.

[2] Richard Kenyon. Tiling a rectangle with the fewest squares. *Journal Combinatorial Theory - Series A*, 76:272–291, 1996.

[3] David A. Klarner. Polyominoes. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, pages 225–240. CRC Press, 1997.

[4] W. T. Trotter, Jr. Tiling bounded open sets with squares that touch the boundary. In *Annals of the New York Academy of Science*, volume 440, pages 304–321. 1985.