Alternating Paths along Orthogonal Segments

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Abstract
It was shown recently that in the segment endpoint visibility graph Vis(S) of any set S of n disjoint line segments in the plane, there is an alternating path of length $\Theta(\log n)$, and this bound best possible apart from a constant factor. This talk focuses on the variant of the problem where S is a set of n disjoint *axis-parallel* line segments, and shows that the length of a longest alternating path in the worst case is $\Omega(\sqrt{n/2})$ and $O(n/2+2)$.

1 Introduction

Given a set S of disjoint line segments in the plane, an *alternating paths* is a simple polygonal path $p = (v_1 v_2, \ldots, v_k)$ such that $v_{2i-1}v_{2i} \in S$, $i = 1, \ldots, \lfloor k/2 \rfloor$ and $v_{2i}v_{2i+1}$ does not cross any segment of S for $i = 1, \ldots, \lfloor (k-1)/2 \rfloor$.

It is known that there are sets of disjoint segments that do not admit an alternating Hamiltonian path. Hoffmann and Töth [3] proved recently, answering a question of Bose [1, 6], that for any set S of n disjoint line segments in the plane, there is an alternating path running through at least $\lceil \log_2(n+2) \rceil - 1$ segments of S, and this bound is best possible apart from a constant factor.

The $O(\log n)$ upper bound construction [6, 3] is a set S of line segments arranged so that every segment $s \in S$ has two endpoints on the convex hull $\mathrm{conv}(\bigcup S)$, and therefore any alternating path containing segments from both sides of s should go through s as well. In that construction n segments have $\Omega(n)$ different orientations. If all segments are axis-parallel, we can prove a better lower bound:

Theorem 1 *For any set S of n disjoint axis-parallel segments in the plane, there is an alternating path running through $\sqrt{n/2}$ segments of S.*

Restricting the general upper bound construction to axis-parallel segments, we obtain an upper bound of $O(n/2+2)$ (see Fig. 1), which leaves room for further improvements from below or from above.

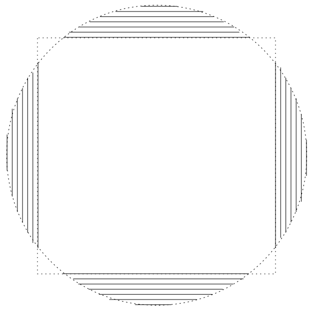

Figure 1: Axis-parallel segments clipped to a disk.

2 Axis-parallel segments

We may assume that there are at least $n/2$ horizontal segments in S. Let H, $H \subseteq S$, denote their set. For two segments $s \in H$ and $t \in H$, we say that s *supports* t if there is an x-monotone and y-monotone curve connecting a point of s to a point of t such that it does not cross any segment of S. (This includes the case where there is vertical visibility between s and t.) We say that $s \prec t$ iff there is a sequence $(s = s_0, s_1, s_2, \ldots, s_r = t)$ in H such that s_i supports s_{i+1}, $i = 0, 1, 2, \ldots, r-1$ (Fig. 2). The relation $\prec$ is a partial order in H. (A similar order was used in [5]).

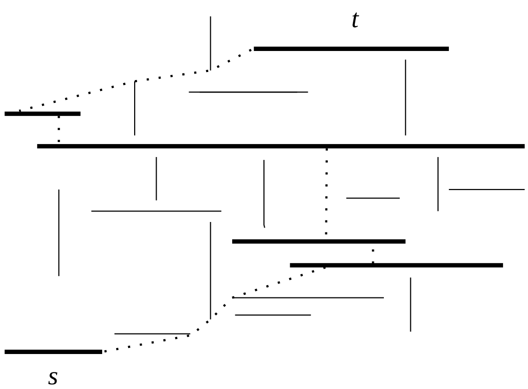

Figure 2: $s \prec t$.

By Dilworth's theorem [2], there is either (i) a chain or (ii) an anti-chain of size $\sqrt{n/2}$ with respect to $\prec$.

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(We note that the size of the maximal chain and anti-chain can be $\sqrt{n/2}$ simultaneously.) In either case, we show that all segments in the chain or anti-chain can be linked together in a common alternating path.

In case (i), let $s_1, s_2, \ldots, s_r$ be a sequence of r, $r \geq \sqrt{n/2}$, segments of H such that each s_i supports s_{i+1}. Denote the left and right endpoint of s_i by a_i and b_i. Let $\gamma(i)$, be the x- and y-monotone curve connecting s_i and s_{i+1} such that the two endpoints of $\gamma(i)$ are $v_i \in s_i$ and $w_i \in s_{i+1}$ (fig. 2).

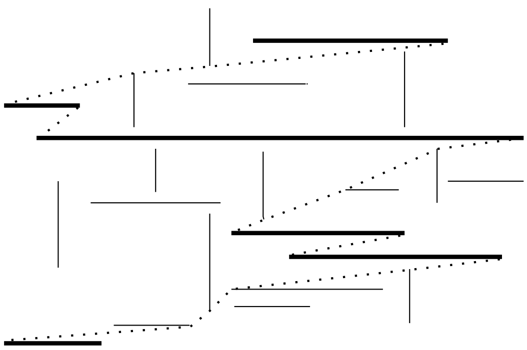

Figure 3: The initial paths $\pi(i)$.

For every i, place a rubber band along the path $(a_i v_i) \cup \gamma(i) \cup (w_i b_{i+1})$. Then let the rubber band contract while its endpoints stay pinned down at a_i and b_{i+1} with the constraint that it cannot cross any segment of S. The rubber band forms a polygonal path $\pi(i)$ through segment endpoints lying between s_i and s_{i+1} (Fig. 4). Notice that $\pi(i)$ remain x- and y-monotone.

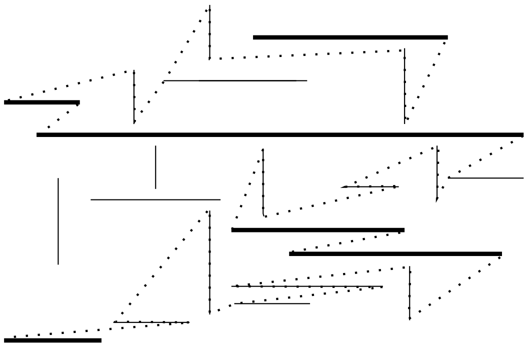

Figure 4: Paths $\pi(i)$ after one iteration.

Next, we expand recursively every $\pi(i)$ into an alternating path between s_i and s_{i+1}. We want to make sure that the concatenation of the resulting $r-1$ alternating paths is also a simple alternating path, that is, the $r-1$ alternating paths are pairwise disjoint.

Consider a path $\pi(i)$. If there is a segment $\hat{s}$ which has exactly one endpoint $v(\hat{s})$ with $\pi(i)$, then we modify $\pi(i)$ to go along $\hat{s}$ and visit the second endpoint of $\hat{s}$. We call this operation an *expansion* of $\pi(i)$. The expansion practically means that we pick the segment of $\pi(i)$ lying before or after the common vertex $v(\hat{s})$ and pull the rubber band to the second endpoint of $\hat{s}$ with the constraint that it cannot cross any other segment of S. We choose the *directions* of the expansion as follows: If $\hat{s}$ is horizontal and lies on the left (right) side of $\pi(i)$ then we expand the segment of $\pi(i)$ above (below) $v(\hat{s})$. If $\hat{s}$ is vertical and lies on the left (right) side of $\pi(i)$ then we expand the segment of $\pi(i)$ to the left (right) of $v(\hat{s})$ (see Fig. 4 and 5).

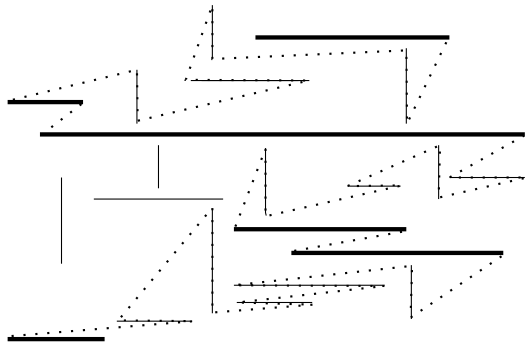

Figure 5: The resulting alternating path.

Inevitably, when we expand a segment of $\pi(i)$ to visit a second endpoint of $\hat{s}$, our path may hit other segment endpoints between s_i and s_{i+1}. The choice of directions of the expansion ensure that $\pi(i)$ never hits a segment endpoint that is already visited by $\pi(i)$ or a second endpoint of a segment whose one endpoint is in $\pi(i)$. We maintain two invariants:

1. Every piece of $\pi(i)$ which does not lie along a segment of S is x- and y-monotone;
2. If an $\hat{s}$ has one common point $v(\hat{s})$ with $\pi(i)$ and lies on the left (right) of $\pi(i)$, then $v(\hat{s})$ is its right or lower (left or upper) endpoint.

One can show that repeating the *expansion* operation on the path $\pi(i)$, we end up with $r-1$ pairwise disjoint alternating paths between the pairs (s_i, s_{i+1}).

In case (ii), note that any two segments in an anti-chain are separated by a vertical line, therefore the segments in the anti-chain have a linear left-to-right order. Consider the r, $r \geq \sqrt{n/2}$, segments of an anti-chain $A = \{s_1, s_2, \ldots, s_r\} \subset H$ labeled according to this order (Fig. 6). Denote the left and right endpoint of each s_i by a_i and b_i.

Observe that $s_i \not\prec s_{i+1}$ and $s_i \not\succ s_{i+1}$ implies that there is no x- and y-monotone curve between

s_i and s_{i+1} which avoids vertical segments but possibly crosses horizontal segments. Specifically, if the y-coordinate of s_i is smaller than that of s_{i+1}, then there is staircase (axis-parallel x- and y-monotone) curve between b_i and a_{i+1} whose every vertical segment is along a vertical segment of S (middle of Fig. 6). Informally, this staircase curve blocks any curve which would imply a relation $s_i \prec s_{i+1}$. (If the y-coordinate of s_i is larger than that of s_{i+1}, there is no specific condition.)

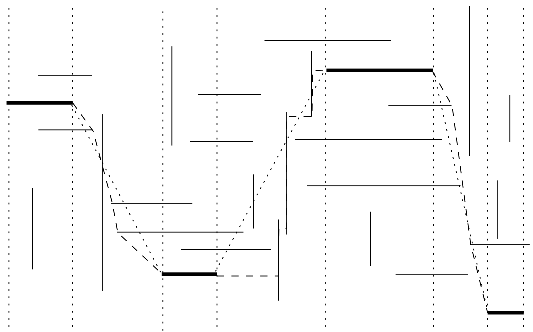

Figure 6: Linear oder in an anti-chain.

For every i, $i = 1, 2, \ldots t-1$, we connect b_i and a_{i+1} by a rubber band $\varrho(i)$: If b_i is above a_{i+1} then we find a monotone descending polygonal path that can cross vertical segments but must avoid horizontal segments. If b_i is below a_{i+1} then consider the polygonal staircase path from b_i to a_{i+1} as described above. In both cases, let us denote this polygonal path by $\varrho(i)$ (dashed in Fig. 6).

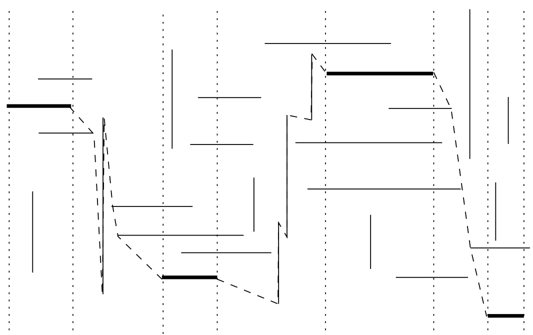

Figure 7: The initial curve $\varrho(i)$ with monotone descending visibility edges.

Recursively, at every intersection point of $\varrho(i)$ with a vertical segment vw, we force the rubber band $\varrho(i)$ to pass through v and w such that $\varrho(i)$ is ascending along vw. In this way, we obtain a curve $\pi(i)$ that does not cross any segment of S and whose pieces not lying along segments of S are x-monotone increasing and y-monotone descending.

Finally, we expand recursively every $\pi(i)$ into an alternating path between b_i and a_{i+1} using the similar expanding operations as in case (i). Consider a segment $\hat{s} \in S$ with one common point $v(\hat{s})$ with $\pi(i)$. If $\hat{s}$ is horizontal and lies on the left (right) side of $\pi(i)$ then we expand the segment of $\pi(i)$ below (above) $v(\hat{s})$. If $\hat{s}$ is vertical and lies on the left (right) side of $\pi(i)$ then we expand the segment of $\pi(i)$ the the left (right) of $v(\hat{s})$ (Fig. 8).

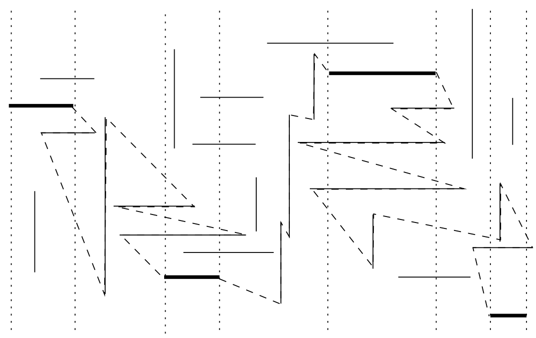

Figure 8: Winding the rubber band around obstacles.

We can maintain similar invariants as in case (i) (now, portions of $\pi(i)$ not lying along segments of S are monotone descending). One can show that the expansion of $\pi(i)$ does not touch any segment endpoint twice, every resulting path is simple. The invariants also ensure that all $r-1$ resulting alternating paths are pairwise disjoint. This completes the proof of Theorem 1.

References

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