Vertex Cover and Connected Guard Set

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The *art gallery problem* asks how many guards are sufficient to see every point of the interior of an n-vertex simple polygon. The guard is a stationary point who can *see* any point that can be connected to it with a line segment within the polygon. A collection of guards $S = \{g_1, ..., g_k\}$ is said to *cover* the polygon P if every point $x \in P$ can be seen by some guard $g \in S$.

For a guard set S we define the *visibility graph* $VG(S)$ as follows: the vertex set is S and two vertices v_1, v_2 are adjacent if the line segment with endpoints v_1 and v_2 is a subset of P, i.e. $\overline{v_1v_2} \subseteq P$. Next, the guard set S is said to be *cooperative* (*connected*) if the graph $VG(S)$ is connected.

The concept of *cooperative guards* was proposed by Liaw, Huang and Lee [5]. They established that *The Minimum Cooperative Guards Problem* for simple polygons is NP-hard, but for spiral and 2-spiral polygons this problem can be solved in linear time [5]. For k-spiral polygons the minimum number of cooperative guards is at most N_k, the total number of reflex vertices in the k-spiral polygon [3]. The cooperative guards problem for general simple polygons has been completely settled by Hernández-Peñalver, proving that $\lfloor \frac{n}{2} \rfloor - 1$ cooperative guards are always sufficient and occasionally necessary to guard a polygon of n vertices [4].

The *diagonal graph* G_D of any triangulation of an n-vertex polygon P is a graph obtained only from $n-3$ internal diagonals of the triangulation: the edges correspond to the diagonals and the vertices correspond to all endpoints of diagonals. Herein we discuss the relation between a vertex cover of a diagonal graph and a connected vertex guard set in a polygon (guards are restricted to be located only at the vertices of the polygon): we show that any set S is a vertex cover of G_D if and only if S forms a connected vertex guard set in P.

1 Diagonal graphs

Lemma 1.1 *Let P, T and G_D be a simple n-vertex polygon ($n \geq 4$), its arbitrary triangulation, and the diagonal graph of triangulation T, respectively. Then G_D is connected.* □

Let us recall that a graph is *outerplanar* if it can be embedded in the plane so that all of its vertices lie on the exterior face.

Lemma 1.2 *Let m be the number of edges of a connected outerplanar graph G. Then there exists a vertex cover of cardinality at most $\lfloor \frac{m+1}{2} \rfloor$.* □

Let P be a polygon of n vertices. Any of its diagonal graphs has $n-3$ edges, and, of course, it is outerplanar. By Lemma 1.1 and Lemma 1.2 we get the following:

Corollary 1.3 *Let G_D be the diagonal graph of a triangulation of an n-vertex polygon. Then there exists a vertex cover of cardinality at most $\lfloor \frac{n-2}{2} \rfloor$.* □

2 Vertex cover vs. connected guard set

A *triangulation graph* G_T of an n-vertex simple polygon P is a graph obtained by triangulation P with internal diagonals between vertices: the vertices of G correspond to the n vertices of P, and the edges correspond to the n edges of P and $n-3$ diagonals.

A *vertex guard* in G_T is a single vertex of G_T. A set of guards $S = \{g_1, \ldots, g_k\}$ is said to *dominate* G_T if every triangular face of G_T has at least one of its vertices assigned as a guard ($\in S$). Finally, the collection of guards $S = \{g_1, \ldots, g_k\}$ is said to be *connected* if for any two guards $g_i, g_j \in S$ there exists a path $p = (g_i, p_1, \ldots, p_l, g_j)$ in triangulation graph G_T that all $p_t \in S$, for $t = 1, \ldots, l$. Guards in graph G_T are called *combinatorial connected guards* to distinguish them from the *geometric connected guards* introduced earlier. The reason for introducing triangulation graphs is that a proof of sufficiency of a certain number of combinatorial connected guards establishes the sufficiency of the same number of geometric connected guards in a polygon.

Lemma 2.1 [4] *Let P be a simple polygon, and G_T be one of its triangulation graphs. If G_T can be dominated by k combinatorial connected guards, then P can be covered by k geometric connected vertex guards.* □

The main use of diagonal graphs is the following result.

Theorem 2.2 *Let T, G_T, G_D be any triangulation of a simple polygon, a triangulation graph of T and the diagonal graph of T, respectively. If $C = \{g_1, \ldots, g_k\}$ is a vertex cover of graph G_D, then C is a connected guard set in G_T.* □

We note in passing that Theorem 2.2 holds also for *iff*:

Theorem 2.3 *Let T, G_T, G_D be any triangulation of a simple polygon, a triangulation graph of T and the diagonal graph of T, respectively. A connected guard set S in G_T is a vertex cover of diagonal graph G_D.* □

Corollary 1.3 and Theorem 2.2 lead immediately to the following:

Corollary 2.4 $\lfloor \frac{n-2}{2} \rfloor$ *connected guards are sometimes necessary and always sufficient to cover any polygon of n vertices.* □

3 Final remarks

The idea of the proof of the sufficiency of $\lfloor \frac{n-2}{2} \rfloor$-bound leads immediately to a linear approximation algorithm AD for finding any connected guard set for a polygon P (guards will be located at vertices):

(1) triangulate P; (in $O(n)$ steps [2])

(2) find any minimum vertex cover of the diagonal graph G_D. (in $O(n)$ steps [8])

Nevertheless, this algorithm can be arbitrarily bad.

Let $S_{AD}(P)$ and $S_{OPT}(P)$ denote the number of connected guards obtained by algorithm AD, and the minimal number of connected guards that cover P, respectively. It is natural to ask about:

$$\lim_{n\to\infty} \max_{G_D} \frac{S_{AD}(G_D)}{S_{OPT}(P)},$$

that is how the obtained result can differ from the optimal solution.

Consider a polygon P of $4k+2$ vertices, its triangulation T, and its corresponding diagonal graph G_D shown in Fig. 1. It is clear, that any minimal vertex cover of G_D is of cardinality $k+1$,

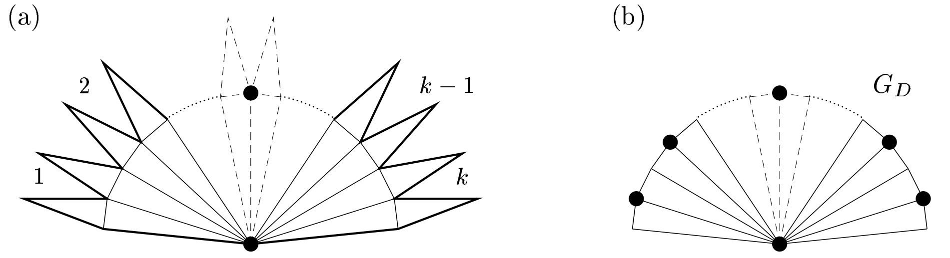

Fig. 1. A star polygon of $4k+2$ vertices that requires only 2 connected vertex guards, (a) its triangulation T, (b) the minimum vertex cover of G_D is of cardinality $k+1$.

and as P is a star polygon with one of its vertices in the kernel, it can be guarded only by two connected vertex guards. Thus:

$$\lim_{n\to\infty} \max_{G_D} \frac{S_{AD}(G_D)}{S_{OPT}(P)} = \infty.$$

We recall that The Minimum Connected Guard Problem for simple polygons was shown to be NP-hard [5].

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