ABOUT VORONOI DIAGRAMS FOR STRICTLY CONVEX DISTANCES *

(EXTENDED ABSTRACT)

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1. Introduction

Voronoi Diagrams in the plane for distances different from the Euclidean one have been considered in several papers[1,2,3,4,5].

Lee[1] has considered this problem for the class of all the L_p-distances for $1 \leq p \leq \infty$, and after studying the behaviour of bisectors, he describes an algorithm, generalizing the standard divide and conquer approach, to construct the Voronoi diagram.

Chew and Drysdale[2] consider the same problem for the more general class of convex distance functions. They propose also the divide and conquer scheme, but do not prove why essential parts of their algorithm, like contour scan during the merge phase, can be applied to convex distance functions as it does to the Euclidean distance.

Klein[3] provides details about a divide and conquer algorithm that works for the class of *nice* distances in the plane. A distance d is *nice* if the following four properties hold: (i) d induces the usual topology. (ii) The d-circles are bounded with respect to the Euclidean distance. (iii) d verifies the *between* condition, i. e. given any two distinct points A and C, there exists a point B, different from A and C and such that: $d(A,C) = d(A,B) + d(B,C)$. (iv) Bisectors are closed sets, homeomorphic to the interval $(0,1)$ and halve the plane in two unbounded regions; moreover it is required that the intersection of any two bisectors has a finite number of connected components.

Whereas the first three properties are fulfilled by every convex distance, it remains open wheather property (iv) always holds.

In this work we provide the first detailed investigation of the class of *strictly convex distances*, their bisectors and their Voronoi regions and study to what extent they look like in the Euclidean case.

First we show that any two d-circles (with respect to a strictly convex distance d) intersect at most twice. Second we prove that for each bisector $Bi(P,Q)$, there exist an homeomorphism mapping the plane onto the plane that sends $Bi(P,Q)$ to a line. As a consequence of the former result, any two *associated* bisectors $Bi(P,Q)$ and $Bi(P,R)$ intersect at most once and, if they do, they cross transversally. This is the reason why the merge step works here as for the Euclidean distance .

But bisectors of a strictly convex distance can behave quite differently from the straight lines of Euclidean distance. For example, bisectors do not always have an asymptotic line. Moreover, there do exist pairs of bisectors $Bi(P,Q)$ and $Bi(R,S)$ that intersect at an infinite number of points. Therefore, strictly convex distances do not in general fulfill property (iv) in the definition of nice distances. However, we show that this problem does not occur if the d-circles are semialgebraic. This is, in fact, the case for all L_p-distances and also for most practical applications.

2. Preliminars

A *strictly convex distance* on the plane is the one induced by any norm and such that the boundary of the unit ball defined by this distance contains no three collinear points. The closure of the unit ball under a strictly convex distance can be characterized as being a compact and strictly convex subset K of the plane that contains the origin as an interior point and that is symmetrical with respect to it. Conversely, any such

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set K can serve to define a normed distance in such a way that the set K is the closure of the unit ball for this distance[6]. The *distance induced by K*, between two points P and Q, is measured as follows: translate K so that it is centered at P and call it K_P. Let Z be the unique point of intersection of the half line from P through Q, with the boundary $Bd\,K_P$ of K_P. Distance between P and Q is the quotient of the Euclidean distances between P and Q and P and Z.

Strictly convex distances verifies the *strong triangle inequality*, i. e.: P does not belong to the closed segment $[X, Y]$ if and only if $d(X,Y) < d(X,P) + d(P,Y)$[5]. Moreover, given two points P and Q, there exists a unique *midpoint* which is the Euclidean midpoint of P and Q[5]. (Given two points P and Q on the plane, a point R is a *midpoint* of P and Q for the distance d if and only if $d(P,R) = d(R,Q) = \frac{1}{2}d(P,Q)$).

Let d be a strictly convex distance on the plane and let P and Q be any two distinct points. The *bisector* $Bi(P,Q)$ of P and Q with respect to the distance d is defined as $Bi(P,Q) = \{X \in R^2 : d(P,X) = d(Q,X)\}$. The *d-circle of centre P and radius r*, $C_d(P,r)$ is defined as $C_d(P,r) = \{X \in R^2 : d(P,X) = r\}$ and equals the boundary $Bd\,B_d(P,r)$ of the open d-ball $B_d(P,r)$ centered at P and of radius r.

2. Summary of Results

Theorem 1. *If d is a strictly convex distance on the plane then any two d-circles intersect at most in two points. As a consequence, given three points on the plane, there exist at most one d-circle containing them.*

Each point X on the bisector $Bi(P,Q)$ of P and Q, is a point of intersection of two, equal radii, d-circles centered at P and Q. This radius can be used to parametrize $Bi(P,Q)$ proving that it is a simple curve topologically like a line as stated in the following Theorem.

Theorem 2. *(topological structure of bisectors). Bisectors for a given strictly convex distance are simple curves that halve the plane in two unbounded regions. Moreover, there exists an homeomorphism from the plane onto the plane, that sends a line onto the bisector.*

Now we are able to give more geometric information about bisectors. Let us introduce some notation. In what follows suppose a strictly convex distance d on the plane is given. Let us call C the unit d-circle. Given any two points P and Q on the plane, let m be the slope of the line determined by the center of C and the point S of contact of one supporting line of C parallel to the line PQ. There is no loss of generality in supposing that line PQ is horizontal and that the midpoint between P and Q is the origin O. Let $r_m(P)$ and $r_m(Q)$ be the lines of slope m through the points P and Q respectively. These two parallel lines determine a *band* of finite width between them.

Theorem 3. *$Bi(P,Q)$ is contained in the band determined by $r_m(P)$ and $r_m(Q)$ and is symmetrical with respect to the midpoint of P and Q.*

We conclude that the *asymptotic direction* of $Bi(P,Q)$ is m. But this doesn't mean at all that an asymptotic line must exist for $Bi(P,Q)$ and even if it exists we only know its slope but not its exact situation. The following theorem gives a necessary and sufficient condition for an asymptotic line for $Bi(P,Q)$ to exist.

Let us introduce first some more notation. Given two distinct points P and Q, consider the d-circle centered in the midpoint of P and Q and passing through P and Q. There is no loss of generality in supposing that this d-circle is the unit circle C and that PQ is horizontal (changing the reference system and scaling if necessary) so the origin O is the midpoint between P and Q. As before let S be the point of contact of the supporting line of C parallel to the line PQ. Note that point S is the highest point in C. Chords $c(h)$ of C parallel to line PQ (i. e. horizontal) at distance h from S are divided in two segments $c_1(h)$ and $c_2(h)$ by the line OS. Let $s_1(h)$ and $s_2(h)$ be their respective lenghts (See Fig.1).

In what follows let us assume that P, Q, C, S, $s_1(h)$ and $s_2(h)$ are as described. With this notation the existence of an asymptotic line for $Bi(P,Q)$ is characterized as follows.

Theorem 4. *A necessary and sufficient condition in order that an asymptotic line for $Bi(P,Q)$ exists is the existence of the following limit:*

$$\lim_{h \to 0} \frac{s_1(h)}{s_2(h)} = l.$$

If limit l exists then the asymptotic line is the one having slope m and passing through a point $T \in [P,Q]$ such that:

$$\frac{\| P-T \|}{\| T-Q \|} = l$$

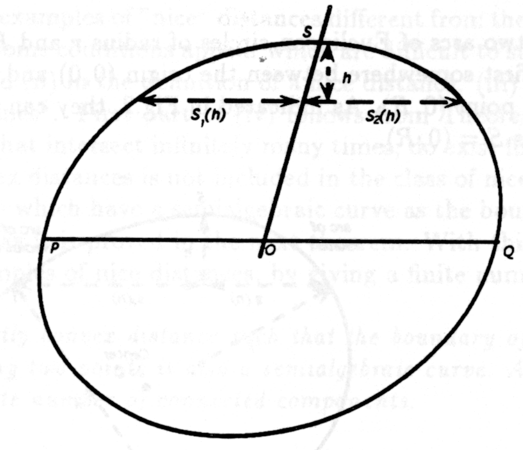

Fig. 1

Remark 1. If an asymptotic line for $Bi(P,Q)$ doesn't exist, then if l_1 (respectively l_2) is: $\underline{\lim}_{h\to 0} \frac{s_1(h)}{s_2(h)}$ (respectively $\overline{\lim}_{h\to 0} \frac{s_1(h)}{s_2(h)}$) we have that the bisector $Bi(P,Q)$ approachs infinitely many times the line l_1 (respectively l_2) of slope m and intersecting segment PQ in a point T (respectively T') such that: $\frac{\|P-T\|}{\|T-Q\|} = l_1$ *(respectively* $\frac{\|P-T'\|}{\|T'-Q\|} = l_2$). This situation implies that $Bi(P,Q)$ must have infinite *inflection points* inside the band determined by the lines l_1 and l_2. Here, by *inflection points* we mean a point in the curve $Bi(P,Q)$ through which there is no line that leaves the curve in one of the halfplanes determined by the line.

Remark 2. Let C and S be as before. Suposse that $f:(x_0-\delta, x_0+\delta) \to R$ is a function such that $f(x_0)=S$ and whose graph equals C in some neighbourhood of S. Note that as f is continuous and strictly convex, the lateral derivatives $f'_+(x_0)$ and $f'_-(x_0)$ always exist. Moreover, as $S=f(x_0)$ is a relative maximum for f, it follows that $f'_+(x_0) \le 0$ and $f'_-(x_0) \ge 0$.

If curve C is not differentiable at S, i. e. if $f'_+(x_0) \ne f'_-(x_0)$, the next Proposition assures that limit l always exists and explicitly gives its value. Let P, Q, C, S, $s_1(h)$, $s_2(h)$ and f as before.

Proposition 1. *If curve C is not differentiable at S, i. e. $f'_+(x_0)$ and $f'_-(x_0)$ are different, then l exists and takes the value:*

$$l = \lim_{h\to 0} \frac{s_1(h)}{s_2(h)} = \frac{1/f'_-(x_0) - 1/m}{1/m - 1/f'_+(x_0)}$$

where m is the slope of line OS, unless $m=\infty$ but then:

$$l = \frac{-f'_+(x_0)}{f'_-(x_0)}.$$

If curve C is differentiable at S, then $f'_+(x_0) = f'_-(x_0) = 0$. It is possible, in this case of differentiability of C at S, that limit l does not exist. But if limit l does exist then the following Proposition can simplify its calculation.

Proposition 2. *If curve C is differentiable at S, then the line OS, which determines the segments $c_1(h)$ and $c_2(h)$, of lenghts $s_1(h)$ and $s_2(h)$ respectively, on the chord $c(h)$ of C, can be replaced by the perpendicular through S to PQ without changing the value of the limit l, i. e.*

$$\lim_{h\to 0} \frac{s_1(h)}{s_2(h)} = \lim_{h\to 0} \frac{p_1(h)}{p_2(h)}.$$

The next example consists on a strictly convex distance d such that any pair of points P and Q, lying on an horizontal line, have a bisector $Bi(P,Q)$ with an asymptotic line not centered in the band determined by the lines $r_m(P)$ and $r_m(Q)$. In the calculation of limit l we will make explicit use of Proposition 2.

Example. Consider two arcs of Euclidean circles of radius r and R, with $r < R$, the second centered in the origin $(0,0)$ and the first somewhere between the origin $(0,0)$ and point $(0,R)$, so that these two arcs have a common tangent at point $(0,R)$. As indicated in Fig.2, they can be considered as part of a unit d-circle C whose highest point is $S = (0,R)$.

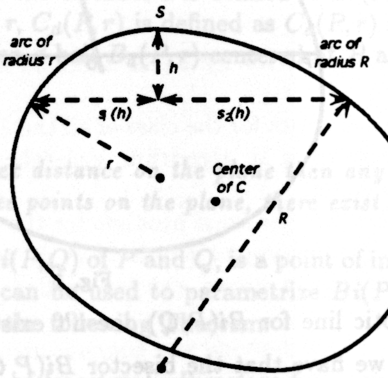

Fig.2

Let us calculate limit l in this case, in order to study the existence of an asymptotic line for bisectors of pairs of horizontal points for such a distance:

$$\lim_{h\to 0}\frac{s_1(h)}{s_2(h)} = \lim_{h\to 0}\frac{\sqrt{r^2-(r-h)^2}}{\sqrt{R^2-(R-h)^2}} = \sqrt{\lim_{h\to 0}\frac{2rh-h^2}{2Rh-h^2}} = \sqrt{\lim_{h\to 0}\frac{2r-h}{2R-h}} = \sqrt{\frac{r}{R}}$$

We conclude that for a pair of horizontal points P and Q and for the distance d induced by such a C, an asymptotic line for $Bi(P,Q)$ exists that is parallel to the line passing through the center O (of C) and S, and such that its point T of intersection with segment PQ verifies the relation:

$$\frac{\| P-T \|}{\| T-Q \|} = \sqrt{\frac{r}{R}}$$

In this case $f'(x_0) = 0$ and $f''_+(x_o) \neq f''_-(x_0)$. Then, as $1/r = f''(x_0)/(1+f'(x_0))^{3/2}$ where r is the radius of curvature, it follows that :

$$\sqrt{\frac{r}{R}} = \sqrt{\frac{f''_+(x_0)}{f''_-(x_0)}}$$

The result in this example suggests that maybe we could, in some cases, eliminate the direct computation of the limit l if we know the second derivatives of curve C at S, in view of the relation between the radii of curvature of curve C at S and the second derivatives of function f at x_0. Moreover limit l can be calculated from the knowledge of the derivatives of the curve C at S, as the following Proposition establishes.

Proposition 3. *If curve C is differentiable of order p at S and the following two conditions hold:*

(i) $f'(x_0) = f''(x_0) = \dots = f^p(x_0) = 0$.

(ii) $f_+^{p+1}(x_0)$ and $f_-^{p+1}(x_0)$ are distinct.

Then:

$$l = \sqrt[p+1]{\frac{f_+^{p+1}(x_0)}{f_-^{p+1}(x_0)}}.$$

Klein and Wood[4] gave no examples of "nice" distances different from the Euclidean one, possibly because in the definition of "niceness" some conditions appear which are difficult to stablish. Note that strictly convex distances verify trivially (i) and (ii) in the definition of a nice distance. (iii) is also verified as strictly convex distances are additive along lines[5]. First part of (iv) follows from Theorem 2. We can show (see the full paper) that pairs of bisectors that intersect infinitely many times, do exist for some strictly convex distances. Thus the class of strictly convex distances is not included in the class of nice distances. However, among the strictly convex distances, those which have a semialgebraic curve as the boundary of the unit ball are nice in the sense of Klein and Wood, as it is proved in the next theorem. With this result we are able to construct many and easy to handle examples of nice distances, by giving a finite number of algebraic conditions.

Theorem 5. *Let d be a strictly convex distance such that the boundary of the unit ball is a semialgebraic curve. Then the bisector of any two points is also a semialgebraic curve. As a consequence the intersection of any two bisectors has a finite number of connected components.*

Though strictly convex distances are in general not nice, they produce, as nice distances do[3], Voronoi diagrams with very good properties. Let us state first the definition of the Voronoi diagram for one of these distances.

Let d be a strictly convex distance on the plane and A a finite collection of points. Let $H(P,Q) = \{X \in R^2 : d(X,P) - d(X,Q) < 0\}$. Then:

$$R_A(P) = \bigcap_{Q \in A-\{P\}} H(P,Q)$$

is *the Voronoi region of P with respect to A* and:

$$Vor_d(A) = \bigcup_{P \in A} Bd\, R_A(P)$$

is *the Voronoi diagram of A with respect to the distance d.*

Theorem 6. *(Properties of Voronoi regions) Let d be a strictly convex distance on the plane and A a finite collection of points. Then:*

(i) $R_A(P)$ is an open and not empty subset of the plane and $R_A(P) = \{X \in R^2 : d(X,P) < d(X,Q)$, for every $Q \in A - \{P\}\}$.

(ii) $R_A(P)$ is star-shaped as seen from P.

(iii) $Cl\, R_A(P) = \{X \in R^2 : d(X,P) \leq d(X,Q)$, for every $Q \in A - \{P\}\}$, where Cl denotes the topological closure.

(iv) $\bigcup_{P \in A} Cl\, R_A(P) = R^2$.

In the Euclidean case bisectors are straight lines and so any two bisectors intersect at most once and transversaly. In the general case of an arbitrarly strictly convex distance d, we just know that bisectors are simple curves and therefore we ask how does the intersection of two bisectors look like. We have already mentioned some wild behaviour of the intersection for some pairs of bisectors. Now we study this problem in the case of *associated* bisectors, i. e. when they are any pair among the bisectors determined by three given points.

Proposition 4. *Let P, Q and R be three points in the plane. Let m and m′ be the asymptotic directions of the associated bisectors Bi(P,Q) and Bi(P,R) respectively. Then:*

(i) The associated bisectors $Bi(P,Q)$ and $Bi(P,R)$ intersect at most once.

(ii) The associated bisectors $Bi(P,Q)$ and $Bi(P,R)$ intersect exactly once if and only if the points P, Q and R are d-cocircular.

(iii) In the case of smooth boundary, the associated bisectors intersect if and only if its asymptotic directions m and m' are different.

(iv) In the case of non smooth boundary, if the asymptotic directions m and m' are different, the bisectors intersect exactly once but nothing can be said if the directions coincide.

We will say that the bisectors $Bi(P,Q)$ and $Bi(P,R)$ associated to three given points P, Q and R intersect *transversaly* if the following two conditions holds:

(a) They intersect (necessarily in exactly one point).

(b) It exists an homeomorphism from the plane onto the plane sending each of the coordinate axes onto each bisector.

The two open regions that $Bi(P,Q)$ determines are, each one, characterized as the set of points where function $f(X) = d(X,P) - d(X,Q)$ is less than (respectively greater than) zero ($\{f < 0\}$ and $\{f > 0\}$). We can say then that $Bi(P,Q)$ divides the plane into two regions each of them having an *associated sign* $<$ or $>$. Note that taking $-f$ instead of f leads to the interchange of signs in the regions.

Similarly, the two open regions that $Bi(P,R)$ determines are characterized by the signs of function $g(X) = d(X,P) - d(X,R)$.

If $Bi(P,Q)$ and $Bi(P,R)$ intersect, then around the intersection point appears a set of regions characterized each of them by a pair of signs, first sign being the one of function f, second sign the one of function g. This pair of signs will be called *a combination of signs*. The possible combinations are $<<$, $<>$, $><$ and $>>$.

Transversality in the intersection of bisectors $Bi(P,Q)$ and $Bi(P,R)$ is equivalent to the appearence of four regions, each of them with one of the possible combination of signs[5]. Non appearence of some combination indicates that the intersection is not transversal and in this case one of the remaining signed regions will be not connected as shown in Fig.3.

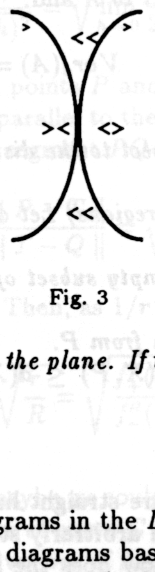

Fig. 3

Theorem 7. *Let P, Q and R be three points in the plane. If their associated bisectors $Bi(P,Q)$ and $Bi(P,R)$ intersect, then they do it transversaly.*

4. References

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