Dominating Polygons by Many Colors
(Abstract)

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1 Introduction

Our aim is to present a new approach to Art–Gallery–type problems. The original Art Gallery Problem raised by V. Klee asks how many guards are sufficient to watch the interior of an n-sided simple polygon.In 1975 V. Chvatal gave the answer proving that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary. Since then many results have been published studying variants of the problem or analyzing algorithmic aspects, see [3],[4] for a detailed discussion. One of the main open questions in this field is the so called Prison Yard Problem for simple rectilinear polygons (comp. [4]), i.e. we want to determine the minimal number of vertex guards sufficient to watch simultaneously both the interior and exterior of any n-sided simple rectilinear polygon. While the case of general simple polygons has been completely settled by Füredi and Kleitman ($\lceil \frac{n}{2} \rceil$ guards for convex and $\lfloor \frac{n}{2} \rfloor$ guards for any non–convex simple polygon are sufficient, see [1]) in the rectilinear case the situation is much worse since their methods seem not to extend as mentioned in their paper and the only upper bound known is the rather trivial $\lfloor \frac{7n}{16} \rfloor + 5$-bound (see [3]) which can be obtained by combining the $\lfloor \frac{n}{4} \rfloor$-result for the interior with the $\lceil \frac{n}{4} \rceil + 1$ guards for the exterior of the polygon.
Below we present two asymptotically tight bounds for the classes of strictly monotone resp. orthoconvex rectilinear polygons. Our main contribution however we see in the way how to prove these bounds as upper bounds. It sheds some more light on the combinatorial nature of the Prison Yard Problem as well as of some other solved and unsolved questions in this field.

2 Lower Bounds

Figure 1 shows an example of a rectilinear polygon due to Dorward who claimed that it required $\lceil \frac{n}{3} \rceil$ guards, see [4]. Continuing, however, periodically the guarding positions indicated in Fig.1 one sees that $\lceil \frac{7n}{24} \rceil + 2$ watchmen are sufficient. Moreover, looking at the most simple possible periodic staircase (Fig.2) a straightforward combinatorial argument shows that $\lceil \frac{3n}{10} \rceil$ guards are necessary and (up to an additive constant) also sufficient as indicated. We will show that this is an upper bound for all strictly monotone rectilinear polygons.

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A pyramid is a rectilinear polygon that has a horizontal edge the length of which equals the sum of the lengths of all other horizontal edges. Fig.3a shows a pyramid P_0 with a guarding set of asymptotic size $5n/16$. We show that it also requires as many guards. The length of the edges of P_0 are chosen in such a way that to watch a quadrilateral inside (Fig.3b) one has to choose one of its vertices as a guard position.

Proposition 1: $\lceil \frac{5n}{16} \rceil$ guards are necessary to watch P_0.

Proof: We distinguish 3 types of guards, comp.Fig.3b. An α–guard pair watches together 4 quadrilaterals, 2 type1–triangles, and 1 type2–triangle. We remark that we can form these pairs since the guard watching 4 quadrilaterals must have at least one "partner" on the other side watching the opposite triangle. β– resp.γ–guards are sitting in concave (convex) corners and they are not part of α–pairs. They watch each 2 (resp.1) quadrilateral, 0 (resp.1) type1–triangle, and 1(resp.1) type2–triangle. Since we have a total of $(n-2)/2$ quadrilaterals and $(n-2)/4$ triangles of each type we conclude for any guarding set consisting of $g = 2a + b + c$ guards of α–, β–, and γ–type respectively that: $4a + 2b + c \geq (n-2)/2$, $2a + c \geq (n-2)/4$, $a + b + c \geq (n-2)/4$. But this implies the lower bound.

3 Upper Bounds

Several upper bound proofs of art gallery problems are based on graph coloring arguments. For example, any triangulation graph of a simple polygon is 3–colorable. Analogously, the graph of any convex quadrilateralization of a simple rectilinear polygon with both diagonals added to each quadrilateral is 4–colorable, see [2]. Then in both cases for any color i the set of all vertices colored with i is sufficient to watch the whole interior of the polygon, since any color dominates any triangle (resp. quadrilateral). Choosing a color class of minimal size, one obtains the $\lfloor \frac{n}{3} \rfloor$–bound for general polygons and the $\lfloor \frac{n}{4} \rfloor$–bound for rectilinear polygons. Unfortunately, this direct graph coloring approach fails for polygons with holes as well as for the prison yard problem.

We start with a graph modelling the prison yard problem for orthoconvex rectilinear polygons. Any such polygon P has 4 extremal edges (northmost, eastmost,...) ordered cyclically (clockwise). Two successive extremal edges are either neighbours or connected by a staircase. The polygon is strictly monotone if there are two disjoint pairs of neighbouring extremal edges. We note that the exterior region of the polygon is covered by the set of quadrants of all stairs and and four halfplanes defined by the extremal edges.

Definition: Let P be a quadrilateralized orthoconvex polygon then $G(P)$ is a graph with a vertex set consisting of all polygon corners and two vertices v and w are connected by an egde if one of the following conditions hold:

- v and w are connected by an edge in the polygon P;
- v and w are connected by a chord of the quadrilateralization;
- v and w are convex corners defining together a stair of a staircase in P.

We say that a subset C of the vertex set dominates the graph $G(P)$ if any quadrilateral, any triangle, and any of the 4 extremal edges contains at least one vertex from C.

Obviously, such a domimating set is sufficient to watch the interior and the exterior of a polygon. Our approach is based on the following idea: Using k different colors we are allowed to label any vertex with a certain number of colored pebbles (this number may depend on the vertex type). If we find a labelling (called k-labelling) such that any color dominates the graph and which uses $f(n)$ pebbles in total then there is a dominating set of size $\left\lfloor \frac{f(n)}{k} \right\rfloor$.

Theorem 2: $\lfloor \frac{3n}{10} \rfloor + 2$ guards are always sufficient to solve the prison yard problem for strictly monotone rectilinear polygons.

Proof: First we remark that for a strictly monotone polygon the dual graph of the quadrilateralization graph is a path, any chord of the quadrilateralization connects a convex with a concave vertex and any quadrilateral has a diagonal connecting two convex vertices (called convex diagonal). There is a greedy algorithm which following the path of the dual graph constructs a 5-labelling of $G(P)$ with the following properties:

- each convex (resp. concave) vertex is labelled by 2 (resp.1) pebbles;
- each triangle and each quadrilateral contains all 5 colors;
- each convex diagonal contains exactly 3 colors, i.e. there is one common color on both sides of the diagonal.

We omit a formal proof and refer to Fig 4. Using 4 more pebbles we can make sure that all extremal edges are also dominated. In total we use $2\frac{n+4}{2} + \frac{n-4}{2} + 4 = \frac{3n+12}{2}$ pebbles.

Applying a similar (but more complicated) algorithm we get 8-labellings of pyramids using $5n + 20$ pebbles. Decomposing a orthoconvex polygon into at most 2 pyramids and one strictly monotone polygon we get the following result.

Theorem 3: $\lfloor \frac{5n}{16} \rfloor + 2$ guards are always sufficient to solve the prison yard problem for orthoconvex rectilinear polygons.

References

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[4] J. O'Rourke, *Computational Geometry Column 15*, SIGACT News 23:2, 1992, pp. 26-28

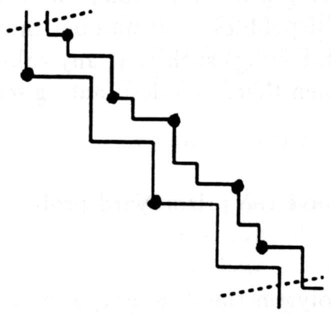

Fig.1

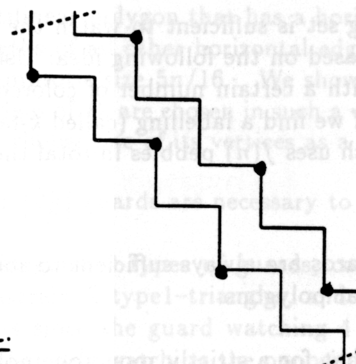

Fig.2

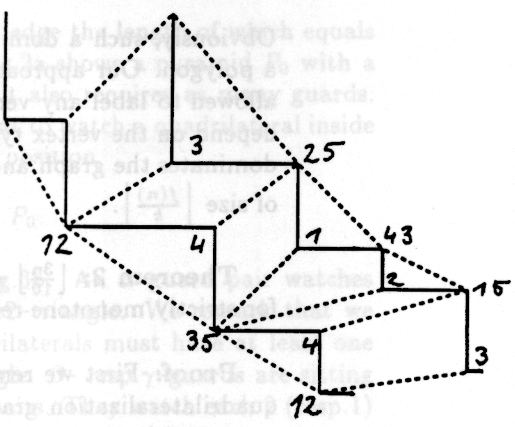

Fig.4

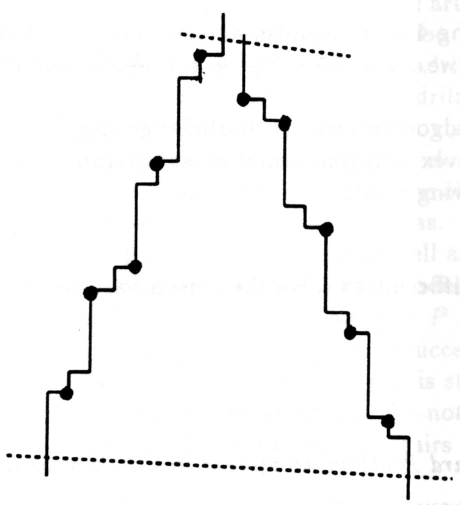

Fig.3a

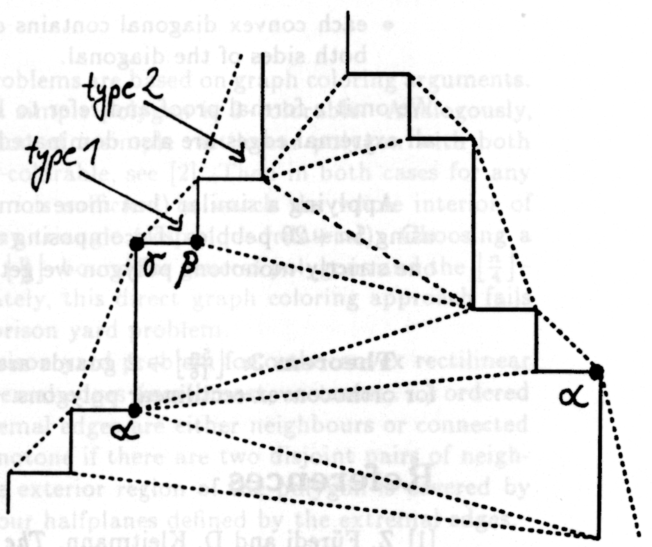

Fig.3b