Circular Visibility*

Jesús García-López, E.U. Informática (U.P. de Madrid)
Pedro Ramos-Alonso, E.U.I.T. Aeronáuticos (U.P. de Madrid)

December 21 1992

Abstract

In this work two problems related with circular visibility are solved. The first one is to compute all the circular movements which take a point that is inside a simple polygon to one of its edges. For this problem, we obtain a linear and thus optimal algorithm. The second one is to compute the region of circular visibility from a point lying inside a polygon. This problem is solved by reduction to the problem of visibility with respect to straight lines in a circular polygon (simple polygon which sides are circular arcs).

1 Introduction

Visibility is one of the most important concepts in Computational Geometry. Until now, the problem considered has been visibility whith respect to straigth lines. For this problem, linear algorithms are given in [EA] for computing the visibility region of a point and for checking whether or not a polygon is visible from one of its edges [AT] and $O(nlog(n))$ time algorithms are given in [LL] and [CG] for computing the visibility region of an edge.

One posible generalization is proposed by Agarwal and Sharir [AS2] and takes us to the concept of circular visibility: two points p and q are circularly visible inside a polygon P if there exists a circular arc from p to q that lies inside P.

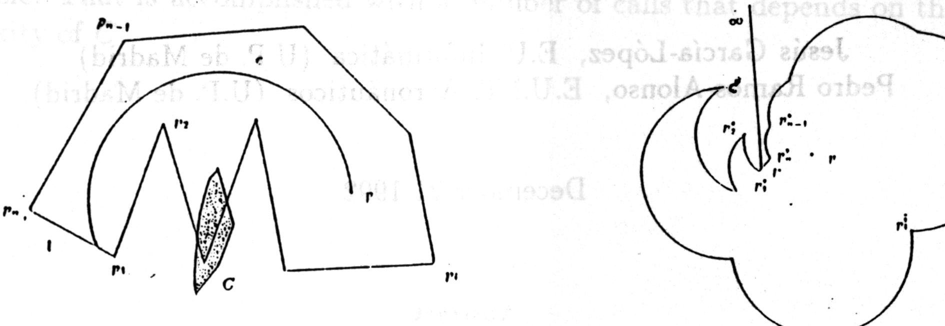
Our work is motivated by the problem of separability [To]. So, from our point of view, circular visibility is, besides an interesting problem by itself, a tool to study circular movements of points and more complicated objects.

In their paper [AS2] Agarwal and Sharir give an $O(nlog(n))$ algorithm to compute the region of circular visibility from a point. We compute the circular movements which take a point p as far as an edge l of the polygon P. We solve this problem in $O(n)$ time with a different technique which can be introduced as follows: We can see that doing an inversion with center at p, the circles passing by p are mapped to straight lines [PS]. This motivates us to use this transformation to, in linear time, reduce the problem of circular visibility to linear visibility.

*This work has been parcially supported by U. Politécnica de Madrid, grant A92011.

2 First Problem

In this section we obtain all the circular movements which take a point p lying inside a simple polygon P as far as one of its edges l. This movements are determined by their centers. So, the solution to the problem will be a region C corresponding to the set of centers (figure 1.a).



a) Set of centers. Figure 1: b) Transformation of a circle c.

Now, we introduce some notation and recall some of the properties of inversion that we will need. The *-image of q will be q^* so by definition of inversion,

$$q^* = \frac{q-p}{\|q-p\|^2} + p .$$

Inversion is an involutive transformation ($q^{**} = q$) and so the map and its inverse are the same.

Remark: Each circular arc between p and l is transformed by the inversion in a half-line between l^* and ∞ (Figure 1.b).

Lemma 1 : *If we take the center of the inversion p as the origin of coordinates, we have:*

1. *The straight line* $r \equiv y - y_0 = m(x - x_0)$ *is mapped to a circle passing by p which center is* $\left(\frac{m}{2(mx_0-y_0)}, \frac{1}{2(y_0-mx_0)}\right)$ *if* $mx_0 - y_0 \neq 0$. *If* $mx_0 - y_0 = 0$ *or* $r \equiv x = 0$ *we have* $r^* = r$ *and if* $r \equiv x = x_0 \neq 0$ *the center is* $\left(\frac{1}{2x_0}, 0\right)$.

2. *The set of lines passing by* (x_0, y_0) *is mapped to a set of circles passing by p and by* $(x_0, y_0)^*$ *which centers are in the bisector of the segment determined by p and by* $(x_0, y_0)^*$.

3. *The set of lines which are tangents to a circle A that passes by p is mapped to a set of circles which centers are on the parabola with focus at p and directrix* A^*.

Remark: We can compute the centers of turns which take p to l computing separately positive (counterclockwise) and negative (clockwise) turns. Moreover, there are positive and negative turns if and only if p and l are linearly visible (we consider straight lines as circles with $r = \infty$).

Next, we see that the sets of centers of positive and negative turns are convex and then we can compute the intersection (and then the union) in linear time [PS]. From now on, we consider only positive turns.

Lemma 2 :

1. The set of centers of positive turns C_l which take a point p to an edge l is convex.

2. The set of points of l which is reached by positive turns is connected.

In the description of the algorithm we consider polygon P described by a list of vertex $p_1, \cdots, p_n, p_1$ traversed counterclockwise and we assume that p_1 and p_n are the vertex of edge l. We denote the boundary of polygon P by ∂P.

Algorithm

Step one: We start from p_1^* and compute the convex chain CC_1^* (figure 2.a) of the simple chain $p_1^*, p_2, \cdots, p_n^*$ (this can be done in linear time with the Graham scan [Gr]) and we obtain the chain CC^* adding l^* to the chain CC_1^*.

Step two: We compute the two support half-lines between each point of l^* and the convex chain CC^* (figure 2.a). In order to do that, we start by p_1^* and traverse the chain in positive sense, dividing l^* into arcs $[a_{i-1}^*, a_i^*]$ $1 \le i \le k_1$ ($p_1^* = a_0^*$ y $a_{k_1}^* = p_n^*$) in such a way that in $[a_i^*, a_{i+1}^*]$ the half-line is supported in the same element w_{1i}^* of ∂P^* (arc of circle vertex). In each step, we must check the relative position of the support half-line and l^*. In the same way, we compute another subdivision of the arc for the other support half-line starting from p_n.

Step three: We merge divisions of step two and obtain division $[c_{i-1}^*, c_i^*]$ $1 \le i \le k$ where the support elements w_i^* y u_i^* are the same in each arc in such a way that the half-cone $u_i^* x^* w_i^*$ with vertex $x^* \in l^*$ satisfies that the angle $u_i^* x^* w_i^*$ is positive and it only contains positive turns (figure 2.b). We obtain the positive turns by checking if some line of the cone passes by p and then we divide the half-cone in positive and negative turns.

Step four: The half-cones with vertex on arcs $[c_{i-1}^*, c_i^*]$ of l^* are the *-images of positive turns with centers in a quadrilateral set S_i (figure 3.a) which sides correspond to a set of lines passing by a point (centers on a segment or half-line) or to a set of lines tangent to a circle (centers on an arc of parabola) (See **lemma 1**).

We observe that arcs $[c_{i-1}^*, c_i^*]$ and $[c_i^*, c_{i+1}^*]$ share the point c_i^* and then the corresponding sets S_i, S_{i+1} share the side that corresponds to the half-cone with vertex in c_i^* (figure 3.a).

According to **lemma 2.2** the set of arcs of the division of l^* is connected and then the union can be computed in linear time.

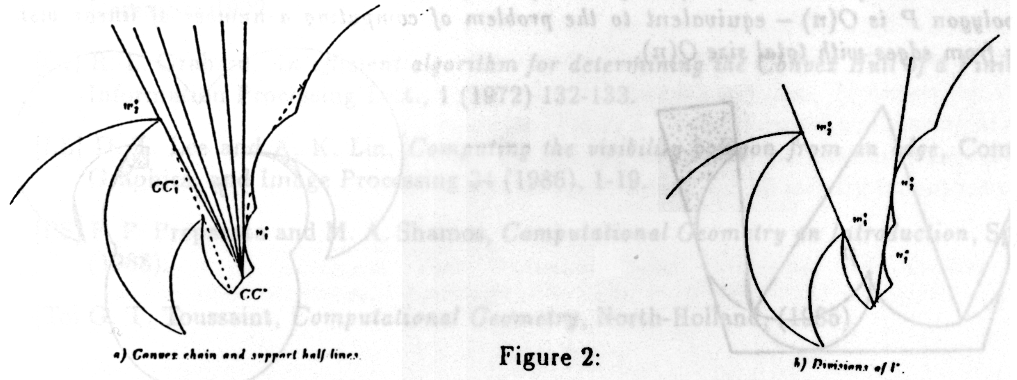
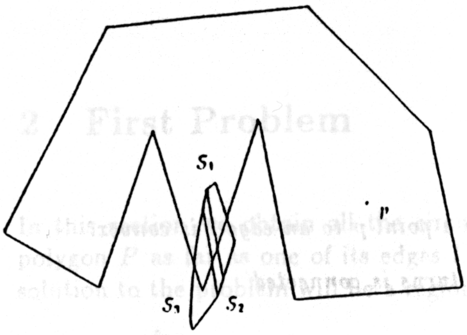


Figure 2:



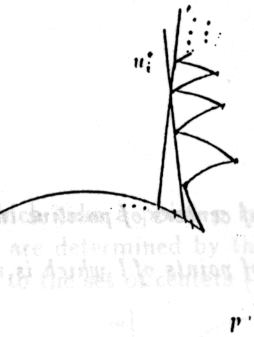


Figure 3:

Each step of the algorithm is carried out in linear time. Observing that region C_1 may have as much as $O(n)$ sides (figure 3.b), we conclude:

Theorem 1 *: Let P be a simple polygon with n edges, p a point in P and l an edge of P. Computing the locus of the centers of turns which take p to l without intersecting P can be done in time $O(n)$ and the algorithm is optimal.*

3 Second Problem

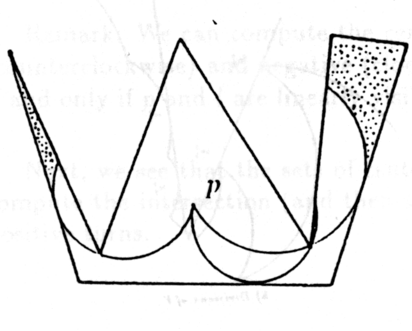
We want to compute the region of circular visibility V of a point p in the interior of a simple polygon P [AS2]. To do that, we use the same technique of inversion at p. The maximal circular paths inside P are half-lines between ∞ and ∂P^*. Then, characterizing the region V is equivalent to characterizing V^* which is the set of points linearly visible from ∞ (figure 4).

We calculate CH^*, the convex hull of the circular polygon ∂P^*, and observe that CH^* generates a number of pockets. Obviously, a point inside a pocket is linearly visible from ∞ if and only if it is visible from the lid of the pocket (figure 4).

Observing that points in the outside of CH^* are visible from ∞, we conclude that computing V^* is reduced to computing linear visibility polygons in each pocket from the correspondent lid.

All the operations involved in transforming one problem to another are linear, so we have proved the next theorem.

Theorem 2 *: The problem of computing the region of circular visibility V of a point p inside a simple polygon P is $O(n)$ − equivalent to the problem of computing a number of linear visibility polygons from edges with total size $O(n)$.*



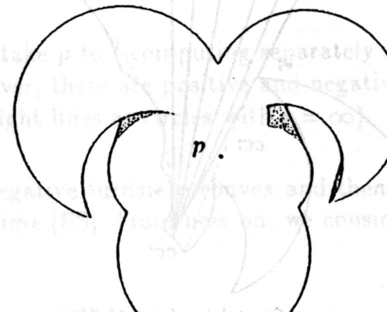


Figure 4:

4 Comments and Open Problems

Going back to the problem which takes us to the study of circular visibility, we obtain a result about separability of points and polygons. The problem consists in checking whether or not a point p outside a polygon P can be taken out the convex hull of P. **Theorem 1** solves this problem in linear time and computes all such turns.

We consider also problems related to circular separability between a circle and a polygon and between a segment and a polygon. The same technique can be applied to the first case [GR] and we are now studying the second one.

Theorem 2 relates concepts of linear and circular visibility and provides an elegant simplification of the problem of circular visibility. This fact increases the interest of the problem of linear visibility from an edge of a polygon. The possibility aforementioned by D. Avis and G. Toussaint [AT] in 1981 of a linear solution to this problem will bring closer the problems of computing linear and circular visibility polygons from a point.

Agarwal and Sharir in [AS1] and [AS2] study some other problems. Among them, how to detect with preprocessing the first intersection of a circle c which passes through a fix point p with the polygon P. A more difficult version, consists of the same problem for circles which do not pass by a fix point. It would be interesting whether or not inversion would simplify the solution of this problems.

5 References

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