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The Geometric Dilation of Finite Point Sets[1]

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Abstract. Let G be an embedded planar graph whose edges may be curves. For two arbitrary points of G, we can compare the length of the shortest path in G connecting them against their Euclidean distance. The supremum of all these ratios is called the *geometric dilation* of G. Given a finite point set, we would like to know the smallest possible dilation of any graph that contains the given points. In this paper we prove that a dilation of 1.678 is always sufficient, and that $\pi/2 = 1.570\ldots$ is sometimes necessary in order to accommodate a finite set of points.

Key Words. Computational geometry, Detour, Dilation, Graph, Network, Spanner, Stretch factor, Transportation network.

1. Introduction. Transportation networks like waterways, railroad systems, or urban street systems can be modeled by a graph G in the plane whose edges are piecewise smooth curves that do not intersect, except at vertices of G.[3]

The quality of G as a means of transport can be measured in the following way. For any two points, p and q, of G, let $\xi_G(p, q)$ denote a shortest path in G from p to q. Then the *dilation* of G is defined by

$$\delta(G) := \sup_{p,q \in G,\ p \neq q} \frac{|\xi_G(p, q)|}{|pq|}. \tag{1}$$

The value of $\delta(G)$ measures the longest possible detour that results from using G instead of moving as the crow flies.

The above definition of $\delta(G)$ does not specify which points p, q of G to consider. There are two alternatives, corresponding to different applications.

Access to a railroad system is only possible at stations. In such a model we would use, as measure of quality, the *graph-theoretic dilation*, where only the *vertices* p, q of G are considered in definition (1). Here, only the lengths of the edges of G are of interest but not their geometric shapes.

Along urban streets, however, houses are densely distributed. Here it makes sense to include *all points* p, q of G in definition (1), vertices and interior edge points alike. This gives rise to the definition of the *geometric dilation* of graph G.

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[3] That is, we do not allow bridges at this stage, but it would, in principle, be possible to enlarge our model.

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The graph-theoretic dilation has been extensively studied in the literature on spanners (see, e.g., Eppstein's chapter in the *Handbook of Computational Geometry* [6] for a survey). One can efficiently construct spanners of bounded dilation and degree, whose weight is close to that of the minimum spanning tree, see [3]. Also, lower time bounds are known, see [4].

In contrast to this, the geometric dilation is a rather novel concept in computational geometry. So far there are only three types of results. Icking et al. [10] and Aichholzer et al. [2] have provided upper bounds to the geometric dilation of planar curves in terms of their oscillation width, and Ebbers-Baumann et al. [5], Agarwal et al. [1], and Langerman et al. [13] have shown how to compute efficiently the geometric dilation of a given polygonal chain or cycle over n edges. Recently, Grüne et al. [7], [8] have given an algorithm for the related problem of computing the detour of a simple polygon. Besides this the geometric dilation was examined in knot theory under the notion of distortion, see, e.g., [11].

In addition to computing the dilation of given graphs, it is quite interesting to construct graphs of low dilation that contain a given finite point set.[4] In the case of graph-theoretic dilation the optimum solution must be a triangulation, since straight edges work best, and adding edges without creating new vertices never hurts. Yet, how to compute the triangulation of minimum graph-theoretic dilation over a given vertex set efficiently seems not to be known. It is not even clear what maximum value the lowest possible dilation over all finite point sets can attain (see Problems 8 and 9 in [6]).

In this paper we address the corresponding question for the geometric dilation. Given a finite point set P, we are interested in the smallest possible geometric dilation of any finite planar graph that contains all points of P, i.e., in the value of

$$\Delta(P) := \inf_{P \subset G,\ G \text{ finite}} \delta(G).$$

Note that, due to the definition of the geometric dilation $\delta(G)$, the points of P may now lie also in the interior of edges. We call $\Delta(P)$ the *geometric dilation of the point set* P. Even for a set P of size 3, computing $\Delta(P)$ is a non-trivial task.

Our main interest in this paper is in the maximal value $\Delta(P)$ can attain, for an arbitrary finite point set P. We prove the following results:

1. There exist finite point sets whose geometric dilation is as large as $\pi/2 = 1.570\ldots$.
2. No finite point set can have a dilation larger than 1.678.

The first result is proven in Section 2, using Cauchy's surface area formula for geometric graphs with cycles and further results for trees. The second result is shown in Section 3. To prove this upper bound we construct a periodic geometric graph G_∞ of dilation $1.6778\ldots$ that covers the plane, such that each finite point set is contained in a slightly perturbed finite part G of a scaled copy of G_∞. While this construction is certainly not efficient—the size of G depends on the rational coordinates of our input set—it serves well in establishing the upper bound.

[4] Observe that the complete graph over P does not solve this problem because the edge crossings would generate new vertices that must also be considered in definition (1).

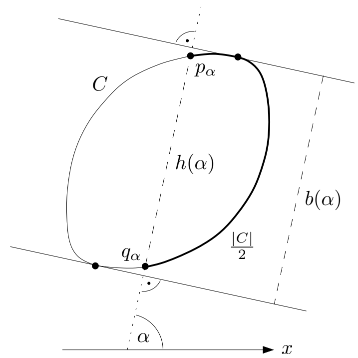

Fig. 1. The breadth of a convex curve is at least its partition distance.

2. A Lower Bound to the Geometric Dilation of Point Sets. In this section we show that some point sets can only be embedded in graphs of large geometric dilation. Our main result is the following.

THEOREM 1. *Let P_n denote the vertex set of the regular n-gon on the unit circle. Then we have $\Delta(P_n) = \pi/2 = 1.570\ldots$ for each $n \geq 5$.*

In order to prove Theorem 1, we will show that neither any graph with cycles nor a tree containing the given point set has a dilation smaller than $\pi/2$. As preparation we prove the following lemma.

LEMMA 1. *Any closed curve C has dilation at least $\pi/2$.*

PROOF. First, let C be a closed convex curve, and let δ denote its dilation. For each direction α, there is a unique pair of points (p_α, q_α), called a *partition pair*, that halves the perimeter $|C|$ of C; see Figure 1. We call

$$h(\alpha) = |p_\alpha q_\alpha|$$

the *partition distance* at angle α. Let $b(\alpha)$ be the breadth of C in orientation α. Clearly, $b(\alpha) \geq h(\alpha)$ holds. Moreover, we have $(|C|/2)/h(\alpha) \leq \delta$, by definition of the dilation.

Thus, by Cauchy's surface area formula,

$$|C| = \int_0^\pi b(\alpha)\, d\alpha \geq \int_0^\pi h(\alpha)\, d\alpha \geq \int_0^\pi \frac{|C|/2}{\delta}\, d\alpha \;=\; \frac{\pi|C|}{2\delta}\ ,$$

hence $\delta \geq \pi/2$.

Next, let C be a closed non-convex curve. Again, for each orientation α there is a partition pair (p_α, q_α) of C. This can be shown by a continuity argument: Clearly, there is a partition pair (p_β, q_β) for *some* orientation β; as we let these points move along C in clockwise direction at equal speed, each will eventually reach its partner's position. During this process, each possible orientation has been attained. Thus, let $h_C(\alpha)$ now denote the *smallest* partition pair distance in direction α, and let $\mathrm{ch}(C)$ denote the convex hull of C. Then $|C| \geq |\mathrm{ch}(C)|$ holds, and we have $b_{\mathrm{ch}(C)}(\alpha) \geq h_C(\alpha)$. So, the proof for the convex case carries over. □

Recently we learned that a different proof of the preceding result was independently given in [11] by Kusner and Sullivan.

Now, let G be an arbitrary geometric graph that contains a bounded face. In this case we provide the following result.

THEOREM 2. *Each graph containing a bounded face has dilation at least* $\pi/2$.

PROOF. Let G be a finite geometric graph in the plane that contains a bounded face. Then there exists at least one cycle in G. Let C be the shortest cycle in G. Then, for any two points, p and q of C, a shortest path $\xi_G(p,q)$ from p to q in G is a subset of C.

Together with Lemma 1 on the dilation of closed curves Theorem 2 follows directly. □

It remains to show that no graph without cycles, i.e., a tree, can provide a smaller dilation for embedding the vertex set P_n of a regular n-gon with $n \geq 5$.

LEMMA 2. *Let tree* T *contain the point set* P_n, $n \geq 5$. *Then* $\delta(T) \geq \pi/2$ *holds*.

PROOF. Assume that tree T contains P_n, and that $\delta(T) < \pi/2$ holds. Then, if p, q are two neighboring points of P_n, the unique path $\xi(p,q)$ in T connecting them is of length at most $\delta(T) \cdot |pq|$, where

$$|pq| = 2\sin\left(\frac{\pi}{n}\right) \leq 2\sin\left(\frac{\pi}{5}\right) = 1.175\ldots,$$

see Figure 2. Therefore this shortest path $\xi(p,q)$ must be contained in ellipse $E(p,q) := \{z \mid |pz| + |zq| \leq (\pi/2)|pq|\}$, i.e., an ellipse with foci p, q and eccentricity $e = 1/(\pi/2)$.

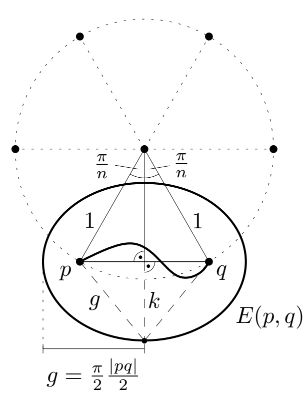

Fig. 2. The path between neighboring points is contained in their ellipse.

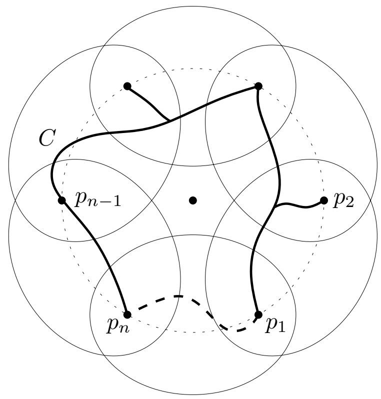

Fig. 3. All paths together, C concatenated with $\xi(p_n, p_1)$, form a cycle.

In such an ellipse $E(p, q)$, per definition, the sum of distances from each point on the boundary of $E(p, q)$ to the foci, p and q, equals

$$|pq| \cdot \frac{\pi}{2} \leq \sin\left(\frac{\pi}{n}\right) \cdot \pi \leq \sin\left(\frac{\pi}{5}\right) \cdot \pi = 1.846\ldots.$$

On the other hand, because both p and q lie on the unit circle, the sum of their distances from the circle's center equals 2. Therefore no ellipse $E(p, q)$ can contain the unit circle's center.

Now we consider the arrangement of all ellipses $E(p_i, p_{i+1})$ of neighboring points, as depicted in Figure 3, and assume that the points are labeled $p_1, p_2, \ldots, p_n$ in counterclockwise order. The concatenation

$$C = \xi(p_1, p_2)\xi(p_2, p_3)\cdots\xi(p_{n-1}, p_n)$$

is a (possibly non-simple) path in T that is contained in, and visits, all ellipses associated with these point pairs. Together with $\xi(p_n, p_1)$, which must be contained in the ellipse of p_n and p_1, C forms a closed path in T that encircles the center of the unit circle and is, thus, not contractible, contradicting the fact that T is a tree. □

Now we can prove Theorem 1.

PROOF. Clearly, for each n we have $\Delta(P_n) \leq \pi/2$, because this is the dilation of the unit circle. Let $n \geq 5$. By Theorem 2 any graph containing a cycle has a dilation $\delta(G) \geq \pi/2$. On the other hand, according to Lemma 2, no tree containing P_n can provide a smaller dilation. So we have shown that no graph containing all points of P_n with $n \geq 5$ can have a dilation smaller than $\pi/2$. □

The arguments contained in the proof of Lemma 2 can also be used in proving the following result.

COROLLARY 1. *Let C be a closed curve, and let P_n be a set of n points evenly placed on C. Then the dilation of any tree containing P_n tends to infinity, as n grows.*

We point out that Theorem 1 does not hold for small values of n. Trivially, $\Delta(P_2) = 1$ holds. For $n = 3$ and $n = 4$ we have the results stated in Corollaries 2 and 3 below. To prove those we first need the following technical lemma.

LEMMA 3. *Let v be a vertex of G where two edges meet at angle α, that is, the tangents to the piecewise smooth edges in the common vertex v form an angle α. Then $\delta(G) \geq 1/\sin(\alpha/2)$ holds.*

PROOF. Let $c_1(t), c_2(t)$ be the unit speed parameterizations of the two edges, satisfying $c_1(0) = c_2(0) = v$. Because all edges are piecewise smooth, c_1 and c_2 are continuously differentiable in an ε-disk centered in v. Therefore we obtain the MacLaurin expansions of c_1, c_2 in v as

$$c_1(t) = \dot{c}_1(0+)t + O(t^2), \qquad c_2(t) = \dot{c}_2(0+)t + O(t^2).$$

Thus, the dilation $\delta(G)$ satisfies the following inequality:

$$\begin{aligned}
\delta(G) &\geq \lim_{t\to 0} \frac{\xi_G(c_1(t), c_2(t))}{|c_1(t) - c_2(t)|} = \lim_{t\to 0} \frac{\xi_G(c_1(t), v) + \xi_G(v, c_2(t))}{|c_1(t) - c_2(t)|} \\
&= \lim_{t\to 0} \frac{2t}{|\dot{c}_1(0+)t - \dot{c}_2(0+)t + O(t^2)|} \\
&= \lim_{t\to 0} \frac{2}{|\dot{c}_1(0+) - \dot{c}_2(0+) + O(t)|} \\
&= \frac{2}{|\dot{c}_1(0+) - \dot{c}_2(0+)|}.
\end{aligned}$$

Simple trigonometry in the equilateral triangle $\triangle(v, \dot{c}_1(0+), \dot{c}_2(0+))$, see Figure 4, yields

$$\sin\left(\frac{\alpha}{2}\right) = \frac{|\dot{c}_1(0+) - \dot{c}_2(0+)|}{2},$$

which proves the claim. □

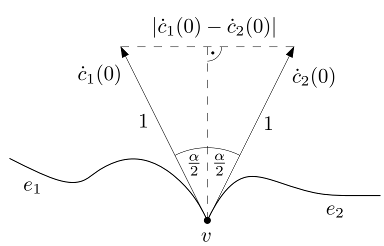

Fig. 4. The angle between tangents of edges in a vertex provides a lower bound on the dilation.

Corollary 2. $\Delta(P_3) = 2/\sqrt{3} = 1.157\ldots.$

Proof. We can achieve the bound $\Delta(P_3) = 2/\sqrt{3}$ by the Steiner tree on P_3, i.e., by connecting the center of the unit circle by a straight segment to each point of P_3. These segments meet at a 120° angle. By Lemma 3, this causes a local dilation of $1/\sin(\alpha/2)$. In our case, no bigger value than $1/\sin(60°) = 2/\sqrt{3}$ can occur.

In order to prove that no graph G containing P_3 can have a smaller dilation we need only consider the following cases. G cannot contain a cycle, because any cycle has a dilation of at least $\pi/2$ according to Lemma 1. Therefore G has to be either a simple chain or a tree with a vertex of degree at least 3. If G is a simple chain passing through the three points in the order p, q, r, then its dilation is at least $(|pq| + |qr|)/|pr| = 2$. Otherwise, G is a tree with a vertex of degree at least 3. However, then we can be sure that its dilation is at least $2/\sqrt{3}$, by the angle argument of Lemma 3. □

Corollary 3. $\Delta(P_4) = \sqrt{2} = 1.414\ldots.$

Proof. Let $p_1, \ldots, p_4$ be the points of P_4 in counterclockwise order. The dilation $\Delta(P_4) = \sqrt{2}$ is obtained by the same type of Steiner tree as above, that is the Steiner tree with one Steiner point s in the center of the unit circle, connected to each p_i by a straight edge; see Figure 5(i). By Lemma 3, the dilation of this tree is not less than $1/\sin(45°) = \sqrt{2}$, and it is easy to show that no greater value than $\sqrt{2}$ occurs.

It remains to show that there is no graph G, having arbitrary, piecewise smooth edges, embedding P_4 with a smaller dilation. Let us assume we have such a graph G. By Lemma 1 we know that G cannot contain any cycle. Consider the shortest paths $\xi_G(p_1, p_3)$ and $\xi_G(p_2, p_4)$. If they do not intersect, at least one of them is longer than $2\sqrt{2}$, see Figure 5(ii), implying $\delta(G) > \sqrt{2}$.

Otherwise there is at least one point of intersection $s \in \xi_G(p_1, p_3) \cap \xi_G(p_2, p_4)$ like that depicted in Figure 5(iii). Because G is a tree, for all $p_i, p_j \in P_4$, $i \neq j$, the shortest path connecting p_i and p_j visits s: $\xi_G(p_i, p_j) = \xi_G(p_i, s) \oplus \xi_G(s, p_j)$. We use this property to complete the proof by showing that $\max\{|\xi_G(p_i, p_{i+1})|\} \geq 2$, where, of course, p_{4+1} denotes p_1.

First, we notice that replacing each $\xi_G(p_i, s)$ by a straight line segment like in Figure 5(iv) does not increase the maximal path length between neighboring points. Now, if s is not located at the unit circle's center, it lies outside of an ellipse $E_2(p_i, p_{i+1}) :=$

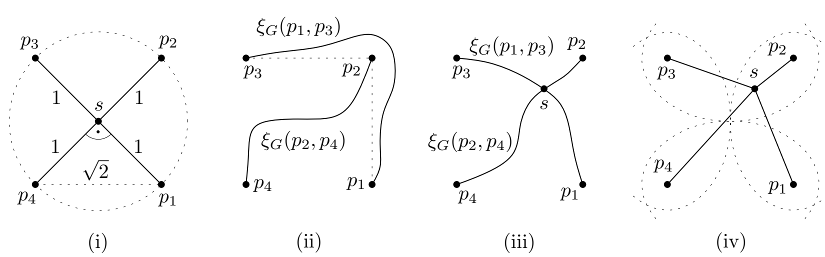

Fig. 5. The Steiner tree of P_4 and the proof of it being optimal.

$\{z \mid |p_i z| + |p_{i+1} z| \leq 2\}$. It follows that

$$\delta(G) \;\geq\; \frac{\max\{|\xi_G(p_i, p_{i+1})|\}}{\sqrt{2}} \geq \frac{\max\{|p_i s| + |p_{i+1} s|\}}{\sqrt{2}} \geq \sqrt{2}. \qquad \square$$

3. An Upper Bound to the Geometric Dilation. In this section we show that each finite point set can be embedded in a finite graph of geometric dilation at most 1.678. More precisely, we prove the following theorem.

THEOREM 3. *There is a periodic plane covering graph G_∞ of dilation $1.67784\ldots$ such that each finite set of points is contained in a slightly perturbed, finite part G of a scaled copy of G_∞.*

We proceed in three steps. First we state a simple yet important technical result in Lemma 4. Then the proof of Theorem 3 starts with the construction of a certain cycle, C. Graph G_∞ will then be obtained by taking the hexagonal grid of unit length, and replacing each vertex with a copy of C. The proof will be concluded by showing how to embed arbitrary finite point sets in G_∞, and by extracting the finite part G.

In determining the geometric dilation of graphs the following lemma is useful; it was first applied to chains in [5]. Observe that an analogous result for the graph-theoretic dilation does not hold.

LEMMA 4. *The geometric dilation of a graph is always attained by two co-visible points.*

PROOF. Assume that $\delta(G)$ is attained by points p, q that are not co-visible and have a minimal Euclidean distance, among all such pairs. Then the line segment pq contains a point r of G in its interior. Hence, we obtain

$$\begin{aligned}\delta(G) = \frac{\xi_G(p,q)}{|pq|} &\leq \frac{\xi_G(p,r) + \xi_G(r,q)}{|pr| + |rq|} \\ &\leq \max\left(\frac{\xi_G(p,r)}{|pr|}, \frac{\xi_G(r,q)}{|pq|}\right) \\ &\leq \delta(G)\end{aligned}$$

by using in step 2 the well-known inequality $(a+b)/(c+d) \leq \max(a/c, b/d)$, for $a, b, c, d > 0$. Thus, the dilation of G is also attained by one of the pairs (p, q), (q, r), a contradiction. $\square$

Now we give the proof of Theorem 3.

PROOF. First we construct a closed cycle, C, that will then be used in building a periodic graph, G_∞, of dilation $\delta(G) = 1.67784\ldots$. The cycle C is a closed curve of constant partition pair distance and somewhat related to the (equi-width) Rouleau triangle. In detail, C is defined as follows. We draw the positive X-axis and two more half-lines

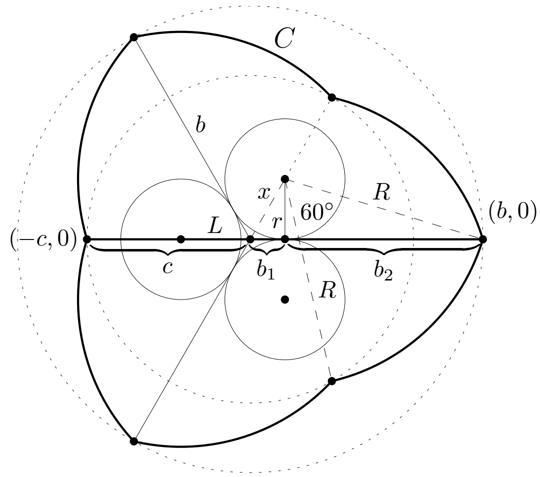

Fig. 6. The cycle C and the essentials of its construction.

starting from the origin at angles of 120° and −120°, correspondingly. Next we fix two numbers, $0 < c < b < 0.5$, that will be specified later, when optimizing the bound. Depending on b and c we draw three circles, each of which touches two of the three half-lines at the distance $b_1 = (b - c)/2$ from the origin; see Figure 6.

For their radius, r, and for the distance, x, from their centers to the origin we obtain

$$\frac{x}{2} = x \cos 60^\circ = b_1 = \frac{b-c}{2},$$

$$\frac{x\sqrt{3}}{2} = x \sin 60^\circ = r,$$

which implies $x = b - c$ and $r = (\sqrt{3}/2)(b - c)$.

Now we consider the line segment, L, of length $b + c$ from $(-c, 0)$ to $(b, 0)$, and imagine that its midpoint is glued to the upper circle. As this circle rotates clockwise by 60°, the right endpoint of L describes a circular arc of length $R\,\pi/3$, where

$$R = \sqrt{r^2 + b_2^2} = \sqrt{\tfrac{3}{4}(b-c)^2 + \tfrac{1}{4}(b+c)^2}$$

$$= \sqrt{b^2 - bc + c^2}.$$

After this rotation, line segment L is unglued from the upper circle, and glued to the left circle instead, that now rotates clockwise by 60°, and so on. This results in a cycle C that consists of six circular arcs of length $R\pi/3$ each.

By construction, each pair of endpoints of the rotating line segment is a partition pair of C, because the endpoints of C are always moving with identical speed. Moreover, each such pair attains the maximum dilation. The argumentation of Lemmata 1 and 2 of [5] shows that there must exist a pair of points attaining maximum dilation which is either a partition pair or contains at least one of the three non-convex corners of C. Finally, Maple analysis shows that the dilation of the partition pairs is dominating.

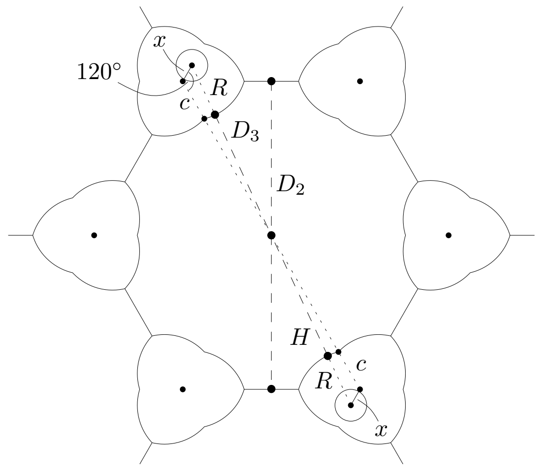

Fig. 7. A cell of the periodic graph G_∞.

So, we have

$$D_1 := \delta(C) = \frac{\pi R}{b+c}.$$

Now we construct a periodic graph G_∞ that covers the plane. This graph is obtained by centering rotated copies of cycle C at the vertices of the regular hexagonal grid[5] of unit edge length, and cutting off those parts of the axes contained in those copies; see Figure 7.

Thanks to Lemma 4, we need only compute the dilation of the two faces occurring in G_∞, which are the cycle C and the boundary of the "dodecagonal" face. On the latter, two candidate partition pairs of points exist that might attain maximum dilation. In the vertical direction we have a pair whose dilation equals

$$D_2 = \frac{2\pi R + 3(1-2b)}{\sqrt{3}}.$$

Observe that the numerator equals three times one-third of the perimeter of cycle C, plus three times the length of a shortened unit edge, whereas the denominator measures the height of the hexagonal cell. The other candidate pair is obtained by intersecting, with the copies of C, the line H connecting the centers of two generating circles; see Figure 7. Since the diagonal of the hexagonal cell is of length 2, the distance between these intersection points equals

$$2(\sqrt{x^2 + 1 - 2x\cos 120^\circ} - R),$$

by the law of cosines. This leads to

$$D_3 = \frac{2\pi R + 3(1-2b)}{2(\sqrt{x^2+x+1} - R)}.$$

[5] Without this refinement, the plain hexagonal grid would have a dilation of $\sqrt{3} = 1.7320\ldots$.

Further analysis shows that the maximum of D_1, D_2, D_3 can be minimized to

$$D_1 = D_2 = D_3 = 1.6778\ldots$$

by putting

$$c = 0.1248\ldots \quad \text{and} \quad b = 0.1939\ldots$$

and that for these values the maximum dilation of C and the "dodecagonal" cycle is indeed attained by the examined candidate pairs.

Now let us assume that we are given a finite set of points,

$$P = \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\},$$

with real-valued coordinates α_i, β_i that must be embedded in a graph of low dilation. We want to scale our modified hexagonal grid G_∞ in such a way that a small perturbation, that affects the dilation of the grid only by an arbitrarily small amount, suffices to accommodate all points of P.

To this end, let γ be the scaling factor for the grid. Then the centers of the horizontal edges we use to embed P have a distance of 3γ in the horizontal direction, and a distance of $\sqrt{3}\gamma$ in the vertical direction; see Figure 8. Moreover, each horizontal edge is of length $(1 - 2b)\gamma$, where b is one of the two radii defining cycle C; see also Figure 6.

Ideally, we would like to place each point of P at the center of some horizontal edge. In general, this is not possible, but the following approximation would be good enough. Let $\eta > 0$ be some given error bound. We want

1. each X-coordinate α_i to be closer to an integer multiple of 3γ than $\eta\gamma$, and
2. each Y-coordinate β_i to be closer to an integer multiple of $\sqrt{3}\gamma$ than $\eta\gamma$.

Indeed, the first condition ensures that point (α_i, β_i) lies almost on the centerline of some column of horizontal edges, that is, it provides a restriction to the horizontal deviation of

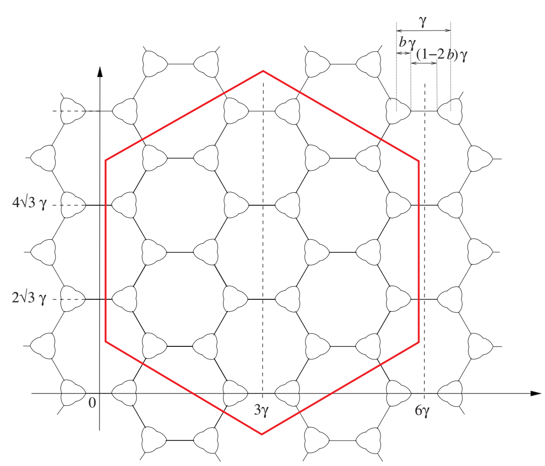

Fig. 8. By Dirichlet's theorem, with slight perturbations on the part of the horizontal edges, G_∞ can embed any finite real point set with dilation 1.678. A finite graph $G \subset G_\infty$ with the same dilation can be obtained by the displayed hexagonal pruning method.

the points. Whereas the second requirement guarantees that the point lies almost on some horizontal edge of this column by restricting its vertical deviation. In both cases the error is arbitrarily small *relative to the grid size*, γ. To visualize this have a look at Figure 8: the periodic infinite graph G_∞ can be constructed such that the n points in the plane to embed lie "close" to the intersection points of the vertical dashed lines $X = 0, \pm 3\gamma, \pm 6\gamma, \ldots$ with the horizontal edges of G they cross. This means that the increase in dilation which might incur by bending n horizontal edges, in order to accommodate the points, can be kept arbitrarily small.

Once the points reside on the modified, infinite grid G_∞ we can cut out a bounded part G containing all embedded points and, for two arbitrary grid points, a shortest path connecting them. If this cut has a hexagonal shape that runs close to the vertex-substituting cycles C, as depicted in Figure 8, the pruning does not increase the dilation.

It remains to prove that the above requirements can indeed be met. We reformulate them as follows. For the given points and the given bound η we want to find integers A_i and B_i and some scaling factor γ such that

$$\left|\frac{\alpha_i}{3}\frac{1}{\gamma} - A_i\right| \le \frac{\eta}{3},$$

$$\left|\frac{\beta_i}{\sqrt{3}}\frac{1}{\gamma} - B_i\right| \le \frac{\eta}{\sqrt{3}}$$

holds. This is possible due to the following well-known approximation result by Dirichlet, see, e.g., [14].

THEOREM 4. *Let $\rho_1, \ldots, \rho_n$ be real numbers, and let $\varepsilon > 0$. Then there exist a natural number q and integers $D_1, \ldots, D_n$ satisfying*

$$|\rho_i q - D_i| \le \frac{1}{q^{1/n}} < \varepsilon.$$

This concludes the proof of Theorem 3 and, thus, of the main result of this section. □

4. Concluding Remarks. In this paper we have, for the first time, studied the geometric dilation of geometric graphs. We have introduced the notion of the geometric dilation, $\Delta(P)$, of a finite set of points, P, as the minimal dilation of all finite graphs that contain P. We have shown that the vertices of the regular n-gon, $n \ge 5$, have dilation $\pi/2 = 1.570\ldots$, and that no finite point set has a dilation greater than 1.678.

These results give rise to many further questions. How can we compute the geometric dilation of a given point set? How costly (in weight and computing time) is the construction of a geometric graph attaining (or approximating) $\Delta(P)$? What is the precise value of

$$\Delta := \sup_{P \text{ finite}} \Delta(P)?$$

(We conjecture $\Delta > \pi/2$.) Finally, what happens if we extend this definition to non-finite sets, e.g., simple geometric shapes?

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