# Inspecting a Set of Strips Optimally

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**Abstract**

We consider a set of axis-parallel nonintersecting strips in the plane. An observer starts to the left of all strips and ends to the right, thus visiting all strips in the given order. A strip is *inspected* as long as the observer is inside the strip. How should the observer move to inspect the set of strips?

**Keywords:** Motion planning, watchman routes

## 1 Introduction

In the last decades, routes from which an agent can see every point in a given environment have drawn a lot of attention (e.g., [3, 4, 5, 6]). Usually, the objective is to find a short route; either the shortest possible route (the optimum) or an approximation. In this paper, we focus on another criterion for routes: We want to minimize the time that a certain area of the environment is *not* seen. Imagine a guard in an art gallery whose objective is to be as vigilant as possible and to minimize the time an object is unguarded. We restrict ourselves to a very simple kind of environments—parallel strips in the plane. More complicated environments are the subject of ongoing research. In Section 2 we present notational conventions and define an objective function which has to be minimized. Then, in Section 3 we first prove some structural properties of an optimal solution for the Euclidean case. At the end we present an efficient algorithm. The ideas can be adapted to the $L_1$-case which is mentioned in Section 4.

## 2 Preliminaries

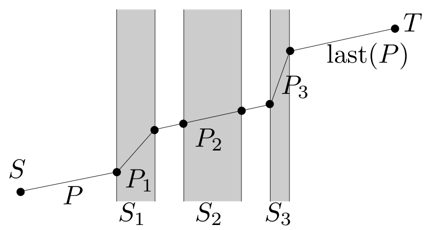


Figure 1: Visiting three strips in a given order.

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Let $\{S_1, \ldots, S_n\}$ be a set of nonintersecting vertical strips and $S = (s_x, s_y)$ be a start point to the left of all strips and $T = (t_x, t_y)$ be an end point to the right of all strips. W.l.o.g. we can assume that $S$ is below $T$ (i.e., $s_y \leq t_y$). Strip $S_i$ has width $w_i$.

An inspection path, $P$, from $S$ to $T$ visits the strips successively from left to right, see Fig. 1. For a given path $P$ let $P_i$ denote the part of $P$ within strip $S_i$. Let $|P_i|$ denote the corresponding path length, and last($P$) the last segment of $P$ (i.e., from $S_n$ to $T$).

While $P$ visits $S_i$, the strip is entirely visible. The performance of $P$ for a single strip $S_i$ therefore is given by $\mathrm{Perf}(P, P_i) := |P| - | P_i|$. The performance of the path $P$ for all strips is given by the worst performance achieved for a single strip. That is $\mathrm{Perf}(P) := \max_i \mathrm{Perf}(P, P_i)$. Finally, the task is to find among all inspection paths the path that gives the best performance for the given situation; that is, an inspection path with minimal performance: $\mathrm{Perf} := \min_P \max_i \mathrm{Perf}(P, P_i)$ .

This problem belongs to the class of LP-type problems [7], but the basis could have size $n$. Therefore, we solve the problem directly. It may also be seen as a *Time and Space* Problem (see, e.g., [2, 1]).

## 3 The Euclidean Case

In this section we first collect some properties of the optimal solution and design an efficient algorithm.

### 3.1 Structural Properties

We can assume that the optimal inspection path is a polygonal chain with straight line segments inside and between the strips: If there are kinks or arcs inside or outside the strips, we can optimize the inspection path by straightening the corresponding parts.

Let us further assume that we have an inspection path as depicted in Fig. 2. The first simple observation is that we can rearrange the set of strips in any nonintersecting order and combine the elements of the given path adequately by shifting the segments horizontally without changing the path length.

Now, it is easy to see that an optimal solution has the same slope between all strips. We rearrange the strips such that they stick together and start from the $X$-coordinate of the start point. The last part of the solution should have no kinks as mentioned earlier.

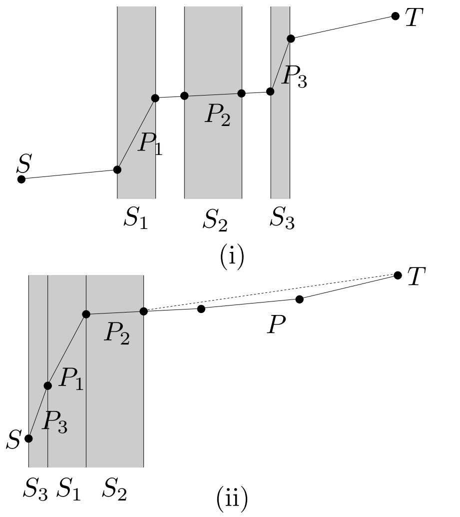


Figure 2: Rearranging strips and path yields the same objective value. Thus, the optimal solution has the same slope between all strips.

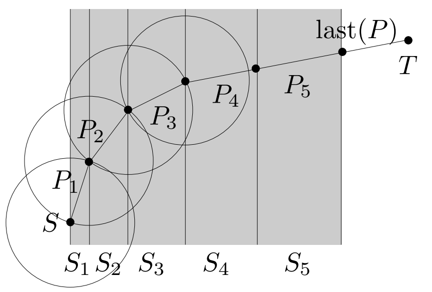


Figure 3: An optimal solution: The first strips are visited with $|P_i| = d$, every other strip with $|P_i| > d$.

**Lemma 1** *The optimal solution is a polygonal chain without kinks between the strips or inside the strips. The path has the same slope between all strips.*

In the following, we assume that the strips are ordered by widths $w_1 \leq w_2 \leq \cdots \leq w_n$, starting at the $X$-coordinate of the start point $s_x$, and lie side by side (i.e., without overlaps or gaps), see Fig. 2(ii). For $t_y = s_y$ the optimal path is the horizontal connection between $S$ and $T$. Thus, we assume $t_y > s_y$.

Now, we show some structural properties of an optimal solution. The optimal solution visits some strips with the same value $d = |P_i|$ until it finally moves directly to the end point, see Fig. 3 for an example.

**Lemma 2** *In a setting as described earlier, the optimal path $P$ visits the first $k \leq n$ strips with the same distance $d$ and then moves directly to the endpoint. That is, for $i = 1, \ldots, k$ we have $|P_i| = d$ and for $i = k+1, \ldots, n$ we have $|P_i| > d$. The path is monotonically increasing and convex with respect to the segment $ST$.*

**Proof.** Let $P$ denote an optimal path for $n$ strips. First, we show that the path is monotonically increasing. There is at least one segment, $P_i$, with positive slope, because $t_y > s_y$. Let us assume—for contradiction—that there is also a segment, $P_j$, with negative slope. We rearrange the strips and the path such that $P_i$ immediately succeeds $P_j$, see Fig. 4(i). Now, we can move the common point of $P_j$ and $P_i$ upwards. Both segments decrease and the performance of $P$ gets better. Thus, there is no segment with negative slope and $P$ is monotonically increasing.

The performance of $P$ is given by $|P_k| := \min_j |P_j|$. Let $k$ be the biggest index such that $P_k$ is responsible for the performance. By contradiction, we show that for $i < k$ there is no $P_i$ with $|P_i| > |P_k|$. So let us assume that $|P_i| > |P_k|$ holds for $i < k$. We can assume $|P_k| = |P_{i+1}|$. From $w_i \leq w_{i+1}$ we conclude that the path $P_i P_{i+1}$ makes a right turn. Because $|P_i| > |P_k|$ we can globally optimize the solution by moving the connection point downward, see Fig. 4(ii). Although $P_i$ increases, the total path length decreases. Thus, $|P_i| = |P_k|$ for $i = 1, \ldots, k-1$. For $i = k+1, \ldots, n$ we have $|P_i| > |P_k|$ by assumption.

Now, we show that there is no kink in the path $P_{k+1}P_{k+2}\cdots P_n$. As $|P_j| > |P_k|$ and $|P_{j+1}| > |P_k|$ we can globally optimize the solution by moving the connection point downwards or upwards, see Fig. 4(iii). The path length decreases. Thus, $P_{k+1}P_{k+2}\cdots P_n$ is a straight line segment.

Altogether, for $i = k+1, \ldots, n$ we have $|P_i| > d$. For $i = 1, \ldots, k$ we have $|P_i| = d$ and the part $P_{k+1}P_{k+2}\cdots P_n \text{last}(P)$ is a straight line segment. Path $P$ is monotonically increasing. The first part of $P$ makes only right turns as already seen. The last part is a line segment. The concatenation of $P_k$ and $P_{k+1}P_{k+2}\cdots P_n$ also makes a right turn; otherwise, we can again improve the performance because $P_{k+1} > P_k$. Altogether, $P$ is convex w.r.t. $ST$. □

In the following, we show that we can compute the optimal solution incrementally; that is, we successively add new strips and consider the corresponding optimal solutions.

Let $S_1, S_2, \ldots, S_n$ be a set of strips and let $S$ and $T$ be fixed. Let $P^i$ denote the optimal solution for the first $i$ strips. For increasing $i$ the parameter $k$ of Lemma 2 is strictly increasing until it remains fixed:

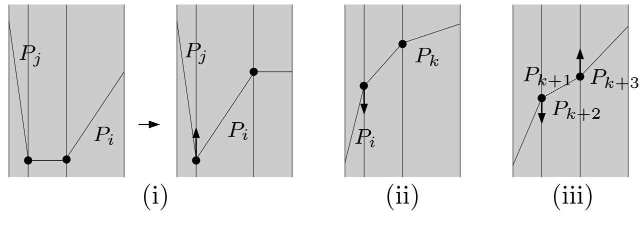


Figure 4: Global optimization by local changes: The connection point can be moved (i) upwards, (ii) downwards or (iii) upwards or downwards.

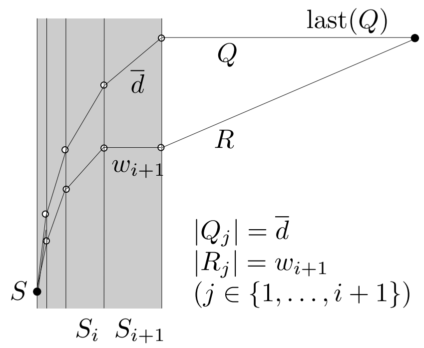


Figure 5: An optimal solution for $k = i+1$ strips can be found between the extremes $R$ and $Q$. $|R_j| = w_{i+1}$ holds and $\overline{d}$ fulfills $|Q_j| = \overline{d}$ with horizontal last$(Q)$.

**Lemma 3** *For $P^i$ either the index $k$ is equal to $i$ or $P^i$ is already given by $P^{i-1}$. If $P^i$ is identical to $P^{i-1}$, also $P^j$ is identical to $P^{i-1}$ for $j = i+1, \ldots, n$.*

We postpone the complete proof of Lemma 3 and first show how to adapt a solution $P^i$ to a solution $P^{i+1}$. Let us assume that we have a solution $P^i$ and that $k$ in the sense of Lemma 2 is equal to $i$. That is, the first $i$ strips are visited with the same distance $d$. If $|P^i_{i+1}| < d$ holds, the solution $P^i$ is not optimal for $i+1$ strips. Therefore we want to adapt $P^i$. Lemma 3 states that it migth be useful to search for a solution with identical path length in $k = i+1$ strips.

We will now show that this solution can be computed efficiently. The task is to compute the path $P^{i+1}$ between two extreme solutions, $R$ and $Q$, as follows: Let $R$ be the path with path lengths $|R_j| = w_{i+1}$ for $j = 1, \ldots, i+1$ and let $Q$ be the path with $|Q_j| = \overline{d}$ for $j = 1, \ldots, i+1$, where the last segment, last$(Q)$, is horizontal. Starting from $d = w_{i+1}$, let $P^{i+1}(d)$ denote the unique path that starts with $|P^{i+1}_j| = d$ for $j = 1, \ldots, i+1$ and ends with a straight line segment.

The performance of $P^{i+1}(d)$ is given by the function $f_{i+1}(d) := di + |\mathrm{last}(P^{i+1}(d))|$. Note that we can express $|\mathrm{last}(P^j(d))|$ in terms of $d$: $|\mathrm{last}(P^{i+1}(d))| = \sqrt{X^2 + \left(t_y - \sum_{j=1}^{i+1}\sqrt{d^2 - w_j^2}\right)^2}$ with $X := t_x - \sum_{j=1}^{i+1} w_j$. By simple analysis, we can show that $f_{i+1}(d)$ has a unique minimum in $d$:

**Lemma 4** *The function $f_{i+1}(d)$ has exactly one minimum for $d \in [w_{i+1}, \delta]$, where $\delta$ is the solution of $t_y - \sum_{j=1}^{i+1}\sqrt{\delta^2 - w_j^2} = 0$ (i.e., the last segment is horizontal).*

Now, it is easy to successively compute the minimum of $f_{i+1}(d)$ for $i = 0, \ldots, n-1$. For example, we can apply numerical methods for getting a solution of $f'_{i+1}(d) = 0$. In the following we assume that we can compute this minimum in time $O(i)$.

Using Lemma 4 we can now prove Lemma 3.

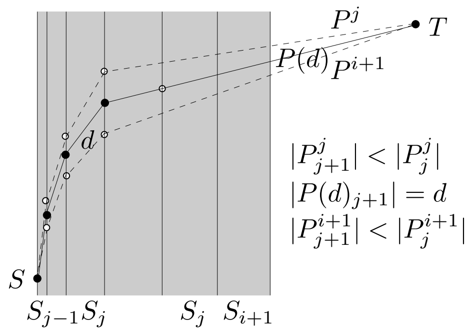


Figure 6: Between $P^{i+1}$ and $P^j$ there has to be a solution with $P(d)$ which is better than $P^{i+1}$.

**Proof of Lemma 3.** Let us assume that we have a solution $P^i$ and let $|P^i_k| = d$, where $k$ denotes the index in the sense of Lemma 2. We use this solution for the first $i+1$ strips. If $|P^i_{i+1}| \geq d$ holds, the given solution is also optimal for $i+1$ strips because the overall performance remains the same: The last segment, last$(P^i)$, of $P^i$ is a line segment with positive slope. Further, $w_j \geq w_{j-1}$ holds. Thus, $P^i$ is the overall optimum; that is, we can apply $P^i$ to all $n$ strips and $|P^i_j| \geq d$ holds for $j = i+1, i+2, n$.

This means that, if we have found a solution $P^i$ with $|P^i_{i+1}| \geq | P^i_k|$ where $k$ denotes the index from Lemma 2, then we are done for all strips.

It remains to show that the index $k$ is strictly increasing until it is finally fixed. From the consideration above we already conclude that there is only one strip, where $k$ does not increase. If $k$ does not increase from $i$ to $i+1$ we have $k < i+1$ and $|P^{i+1}_k| < |P^{i+1}_{i+1}|$. Thus, $P^{i+1}$ is the overall optimum and $k$ is fixed.

Finally, we show that indeed $k$ can never decrease. Let us assume from $\ell = 1$ to $\ell = i$ we have a solution $P^\ell$ for $\ell$ strips and $k = \ell$ for every $P^\ell$. Let us further assume that for $i+1$ strips the solution $P^{i+1}$ comes along with $k = j < i$. We compare the two solutions $P^{i+1}$ (with $k = j < i$ and $|P^{i+1}_j| =: d_{i+1}$) and $P^j$ (with $k = j$ and $|P^j_j| =: d_j$), see Fig. 6.

As $P^j$ is optimal for $j$ strips but not for $j+1$, we have $|P^j_{j+1}| < d_j$. On the other hand $P^{i+1}$ is optimal for $i+1$ strips; thus $|P^{i+1}_{j+1}| > d_{i+1}$ holds; see Lemma 2.

Now for a parameter $d$ consider a monotone path $P(d)$ that starts from $S$, has equal path length $d$ in the first $j$ strips and then moves toward $T$. While $d$ increases, the slope of the last segment strictly decreases. Therefore, the path length $|P(d)_{j+1}|$ is strictly decreasing in $d$. This means that $d_j > d_{i+1}$ and $P^j$ runs above $P^{i+1}$. The path $P(d)$ changes continuously, therefore in $[d_{i+1}, d_j]$ there has to be a value $d'$ such that $|P(d')_{j+1}|$ is equal to $d'$. The path $P(d')$ runs between $P^{i+1}$ and $P^j$. Obviously, $d' < P(d')_l$ for $l = j+1, \ldots, n$ holds.

We show that $P(d')$ is better than $P^{i+1}$. This is a direct consequence of Lemma 4. The value of $P(d)$ strictly decreases from $d = d_{i+1}$ to the unique minimum $d = d_j$. Altogether, $P^{i+1}$ is not optimal. □

The result of Lemma 3 now suggests a method for computing the optimal path efficiently. Starting from $j = 1$ we compute an optimal path for the first $j$ strips. Let $P^j$ denote this path. If $|P^j_{j+1}| > |P^j_j|$ holds we are done. Otherwise, we have to compute $P^{j+1}$ for $j+1$ strips and so on.

### 3.2 Algorithm and its Analysis

**Theorem 5** *For a set of $n$ axis-aligned strips the optimal inspection path can be computed in $O(n \log n)$ time and linear space.*

**Proof.** First, we sort the strips by width which takes $O(n \log n)$ time. Then we apply binary search. That is, in a first step we compute a solution with respect to $j = \lfloor \frac{n}{2} \rfloor$ strips. This can be done by computing the best value for $f_j(d)$ starting from $w_j = d$ until the last segment is horizontal, see Lemma 4. Let $d_j$ denote the optimal value for $j$ strips and $P^j$ the optimal path.

Now, we have to determine whether the optimal path visits $i \leq j$ or $i > j$ strips with the same distance $d_i$. If $d_j > w_{j+1}$, we have to take into account at least the strip $S_{j+1}$. Therefore, $i > j$ holds and we proceed recursively with the interval $[j, n]$. If $d_j \leq x_{j+1}$ we proceed with the interval $[1, j]$. Therefore we will find the optimum in $\log n$ steps. Computing the minimum of $f_j(d)$ for index $j$ takes $O(j)$ time. □

For a lower bound construction we can simply assume that the input of an algorithm is given by an unsorted set of strips. The $X$-coordinates of the strip's left boundaries and their widths describe the setting. The solution is given by a polygonal chain from left to right representing the order of the left boundaries. Thus, sorting a set of $n$ elements can be reduced to the given problem.

**Theorem 6** *For a set of $n$ axis-aligned unsorted strips the optimal inspection path is computed in $\Theta(n \log n)$ time and $\Theta(n)$ space.*

## 4 The $L_1$-Case

Fortunately, if we measure the distance by the $L_1$ metric the structural properties are equivalent. Computing the optimal path becomes much easier.

Note that the path segments between the strips have to be horizontal. We have to distribute the vertical distance from $S$ to $T$ among a subset of the strips. Again we sort the strips by their widths and rearrange the scenario. Fig. 7 shows an example of an optimal $L_1$-path after rearrangement.

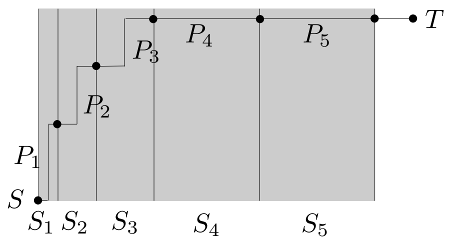


Figure 7: An optimal solution in the $L_1$-case after rearrangement. The first three strips are visited with the same value $d$, for all other strips $|P_i| > d$ holds. The path is horizontal between strips.

**Theorem 7** *The optimal $L_1$-path $P$ visits the first $k \leq n$ strips with the same $L_1$-distance $d$ and then moves horizontally to the end point $T$. For $i = 1, \ldots, k$ we have $|P_i| = d$ and for $i = k+1, \ldots, n$ we have $w_i > d$. Additionally, $\sum_{i=1}^{k} |P_i| = t_y$ holds. If the number of strips, $n$, increases, the index $k$ increases until it remains fixed. The optimal path can be computed in $\Theta(n \log n)$ time and $\Theta(n)$ space.*

An algorithm for the $L_1$ problem is given as follows. First, we sort the strips by their widths. Then starting from $i = 1$ we distribute $t_y + \sum_{j=1}^{i} w_j$ among $i$ strips. For an optimal path, $P^i$, for $i$ strips we have $|P^i_j| = \frac{1}{i}(t_y + \sum_{j=1}^{i} w_j)$ for $j = 1, \ldots, i$. If $|P^i_i| < w_{i+1}$ this path is also optimal for $i+1$ (and $n$) strips. For $|P^i_i| > w_{i+1}$ we distribute $t_y + \sum_{j=1}^{i+1} w_j$ among $i+1$ strips; that is, $|P^{i+1}_j| = \frac{1}{i+1}(t_y + \sum_{j=1}^{i+1} w_j)$ for $j = 1, \ldots, i+1$.

Altogether, if the strips are given by ordered widths, the algorithm runs in $\Theta(n)$ time and space.

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