On the Optimality of Spiral Search

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Abstract

Searching for a point in the plane is a well-known search game problem introduced in the early eighties. The best known search strategy is given by a spiral and achieves a *competitive ratio* of 17.289... It was shown by Gal [14] that this strategy is the best strategy among all *monotone* and *periodic* strategies. Since then it was unknown whether the given strategy is optimal in general. This paper settles this old open fundamental search problem and shows that spiral search is indeed optimal. The given problem can be considered as the continuous version of the well-known m-ray search problem and also appears in several non-geometric applications and modifications. Therefore the optimality of spiral search is an important question considered by many researchers in the last decades. We answer the logarithmic spiral conjecture for the given problem. The lower bound construction might be helpful for similar settings, it also simplifies existing proofs on classical m-ray search.

Keywords: Search games, computational geometry, motion planning, spiral search, m-ray search, competitive analysis, lower bound

1 Introduction

Search games (i. e., games where two players, a searcher and a hider, compete with each other) are studied in many variations in the last 60 years since the first work by Koopman in 1946. For example, Bellman [5] introduced the search for an immobile hider located on the real line with a known probability distribution, Gal [14] and independently Baeza-Yates et al. [2] solve this problem for a uniformly distributed location of the hider. The book by Gal [14] and the reissue by Alpern and Gal [1] gives a comprehensive overview on results on search games.

The length of the searcher's trajectory is often used as payoff of a search game. To get a finite value for the game, we use the competitive framework, that is, we compare the length of the searcher's trajectory to the shortest distance to the hider. Gal [14] calls this a normalized cost function. More precisely, we call a search strategy *competitive* with a factor C, if $|\Pi| \leq C \cdot |\Pi_{\text{opt}}|$ holds for every location of the hider, where $|\Pi|$ denotes the length of the searcher's path and $|\Pi_{\text{opt}}|$ the shortest path to the hider. For analysing the efficiency of a search startegy we use the competitive framework which was introduced by Sleator and Tarjan [28], and used in many settings since then, see for example the survey by Fiat and Woeginger [10] or, for the field of online robot motion planning, see the surveys [24, 17].

We consider a well-known search game problem introduced by Gal [14], namely *searching for a point in the plane*. Starting from a fixed origin O we move along a path Π through the plane. Let us assume that there is an unknown target point t and let p_t denote the first point on Π so that t lies on the line segment between O and p_t. We detect t at point p_t. This means, that we *sweep* the plane until finally the unknown target t is found, see Figure 1.

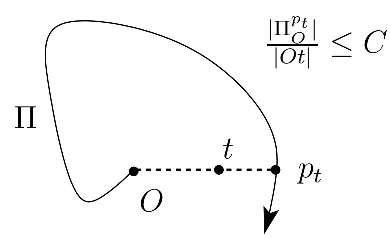

Figure 1: Searching for a point in the plane.

The efficiency of the search path Π is given by the worst-case target, $C := \sup_t \frac{|\Pi_O^{p_t}|}{|Ot|}$, the constant C is called the *competitive ratio* of Π. It was shown by Gal [14] that a spiral strategy is the best strategy among all *monotone* and *periodic* strategies. A strategy S represented by its radius vector $X(\theta)$ is called periodic and monotone, if θ is always increasing and X also satisfies $X(\theta + 2\pi) \geq X(\theta)$. Gal states that it might be a *complicated task* to show that there is a periodic and monotone optimal strategy, a lower bound remains open.

The given problem can be considered as the continuous version of the well-known m-ray search problem which in turn appears in several modifications, see for example [3, 4, 19, 16, 20, 22]. The problem was first introduced and solved by Gal [13]. Namely Schuierer and

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López-Ortiz intensively studied and solved many variants of discrete m-ray search problem, see [16, 21, 25, 26, 22, 27, 23]. The paradigm of *doubling* has also many applications in non-geometric areas, see the overview of Chrobak and Kenyon-Mathieu [7]. Additionally, there are continuous versions of the problem where an optimal solution is given, if the logarithmic spiral conjecture can be proven, see [14, 2, 11, 12, 9]. For example Finch and Zhu considered the problem of searching for a line in the plane, the *relevant conjecture that the family of logarithmic spirals contains the minimal path remains open*, see [12].

Altogether, optimality of spiral search is an important questions considered by many researchers in the last decades and lower bounds are very difficult to achieve.

The paper is organized as follows. In Section 2 we present the best spiral strategy given in [1] and in Section 3 we show that this strategy is indeed optimal in general. The lower bound construction makes use of some transformations from the continuous problem to a *discrete* version. A given strategy is applied to a special 2-ray version of the problem and the best performance for this problem gives a lower bound on the original problem.

Another breakthrough is that for the 2-ray version we can make use of a *visiting order* of rays induced by the *smallest-current-depth* rule. This idea simplifies also many other proofs in the m-ray search setting. Note that an analogous rule was proven to be optimal by López-Ortiz and Schuierer in the presence of a parallel search by p agents, see [23]. Apparently, by the same idea one can easily show that there is an optimal strategy in the classical m-ray setting that is periodic and monotone. Therefore the given lower bound construction gives some more insight in the structure of such problems and it can be applied for variants of the problem.

In the following, we will assume that the target point is at least one step away from the start. Otherwise, we use an additive constant A which means $|\Pi| \leq C \cdot |\Pi_{\text{opt}}| + A$, see [28]. These two settings are equivalent.

2 Spiral search

We consider a logarithmic spiral, Π, which is given in polar coordinates by $(\varphi, e^{\varphi \cot(\alpha)})$ for $-\infty < \varphi < \infty$, see Figure 2 for an example with $\alpha \leq \pi/2$. The angle $\alpha \leq \pi/2$ expresses the excentricity of the spiral. The length of the spiral from the center O to some point q is given by $\frac{1}{\cos\alpha}|Oq|$, for details see [6]. We denote this path by Π_O^q and its length by $|\Pi_O^q|$.

The spiral expands successively and for every target point t there will be a first point p_t on the spiral so that the segment p_tO will hit the target t for the first time. Obviously, the worst case for the competitive ratio is given, if we miss the target t arbitrary close to $(\varphi, e^{\varphi \cot\alpha})$ and detect t at $p_t = (\varphi + 2\pi, e^{(\varphi+2\pi)\cot\alpha})$, see Figure 2. Altogether, the worst-case ratio for the spiral is given by

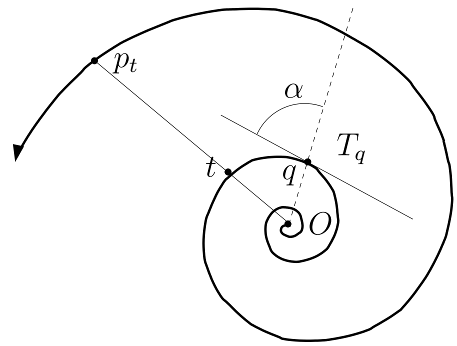


Figure 2: A logarithmic spiral and the worst-case situation.

$$\frac{|\Pi_O^{p_t}|}{|Ot|} = \frac{1}{\cos\alpha} e^{2\pi\cot\alpha}\,.$$

Therefore for $b := \cot\alpha$ we have to find $\min_b e^{b\,2\pi}\sqrt{1+\frac{1}{b^2}}$ which gives $17.289\ldots$, the minimum is achieved for $b = 0.15540\ldots$ This strategy is the best strategy among all *periodic and monotone* strategies, see Alpern and Gal [1].

THEOREM 2.1. *The optimal spiral for searching a point in the plane achieves a competitive ratio of* $17.289\ldots$

3 Lower bound construction

Let us first consider a discrete version of the problem using a bundle of m rays that emanate from the origin and which are separated by an angle $\alpha = \frac{2\pi}{m}$, see Figure 3. The target will be on one of the rays. Again, the goal is detected, if it is swept by the radius vector of the trajectory, i.e., t is hit by a segment p_tO and p_t is visited on the corresponding ray.

Note that any strategy for the continuous version of the problem gives always a strategy for the discrete version. Therefore any lower bound for the discrete version will give a lower bound for the continuous version. Up to now we can neither assume that we have to visit the rays in a periodic order nor that the depth of the visits increases in every step.

We represent an infinite search strategy, S, as follows: In the ith step, the searcher hits a ray—say ray l—at distance x_i from the origin, moves a distance $\beta_i x_i - x_i$ along the ray l, and leaves the ray

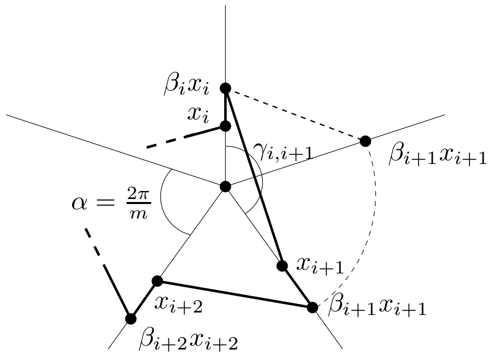


Figure 3: A bundle of rays, a reasonable strategy and a shortcut.

at distance $\beta_i x_i$ with $\beta_i \geq 1$. Then, within length $\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1}\cos\gamma_{i,i+1} + x_{i+1}^2}$ (by the law of cosine) it moves to the next ray at distance x_{i+1}, see Figure 3. Note that any search strategy for our problem can be described in this way. Let us assume that the ray l was visited up to distance $\beta_k x_k$ and is visited the next time at index J_k with distance x_{J_k} and the strategy plans to examine points beyond $\beta_k x_k$. A goal below $\beta_k x_k$ was detected earlier. We can assume $x_{J_k} \geq \beta_k x_k$ by the following argument. For $x_{J_k} < \beta_k x_k$ the strategy has to slip along ray l in order to detect points beyond $\beta_k x_k$. By the triangle inequality a direct movement to $\beta_k x_k$ is more efficient.

The worst case occurs if the searcher slightly misses the goal while visiting ray l up to distance $\beta_k x_k$. Instead, it finds the goal at step J_k and distance x_{J_k} on ray l and the goal is arbitrarily close to $\beta_k x_k$. Altogether, the competitive ratio, $C(S)$, is given by

$$\sup_k \frac{\sum_{i=1}^{J_k-1} \beta_i x_i - x_i + \sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1}\cos\gamma_{i,i+1} + x_{i+1}^2}}{\beta_k x_k} \tag{3.1}$$

Note that the goal is at least one step away from the start. Therefore we set $\beta_0 x_0 = \beta_{-1}x_{-1} = \ldots = \beta_{-m+1}x_{-m+1} = 1$ for initializing the first visits on the rays.

We simplify the problem for m rays in some steps. We do not change the movement of the strategy but we will improve the ratio (3.1) for every k. Instead of the distance

$$\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1}\cos\gamma_{i,i+1} + x_{i+1}^2} + \beta_{i+1}x_{i+1} - x_{i+1}$$

from $\beta_i x_i$ to x_{i+1} and then to $\beta_{i+1}x_{i+1}$ between two arbitrary successive rays we let this distance *shrink* to $\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i \beta_{i+1} x_{i+1}\cos\frac{2\pi}{m} + (\beta_{i+1}x_{i+1})^2}$. This would be the distance by the law of cosine between two neighboring rays without slipping along the second ray, see the dashed line in Figure 3.

The new distance is obviously not greater than the original one, because $\gamma_{i,i+1} \geq \frac{2\pi}{m}$ holds and by triangle inequality $\sqrt{(\beta_i x_i)^2 - 2\beta_i\beta_{i+1}x_i x_{i+1}\cos\gamma_{i,i+1} + (\beta_{i+1}x_{i+1})^2}$ is not greater than $\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1}\cos\gamma_{i,i+1} + x_{i+1}^2} + \beta_{i+1}x_{i+1} - x_{i+1}$. Of course it might be the same for $\beta_{i+1} = 1$ and two neighboring rays, which means $\gamma_{i,i+1} = \frac{2\pi}{m}$. We change only the path length for the ratio but we do not change the movements of the given strategy. Therefore the ratio (3.1) cannot increase, because the numerator will not increase. We additionally set $\beta_1 x_1 = x_1$ for the starting point which will also not increase the numerator.

There is only one problem in the above reformulation concerning the last value of the sum in the numerator of ratio (3.1). The last step of the strategy (before detecting the goal at $\beta_k x_k$) goes from $\beta_{J_k-1}x_{J_k-1}$ to x_{J_k} and not directly to $\beta_{J_k}x_{J_k}$ and this step might indeed be smaller than the distance from $\beta_{J_k-1}x_{J_k-1}$ to $\beta_{J_k}x_{J_k}$. Therefore we simply omit this last step which will decrease the ratio again a bit. One can imagine that this last step will have no influence if we let m go to infinity at the end. Again, we do not change the movement of the strategy, we only improve the ratio.

For convenience, from now on we denote the values $\beta_i x_i$ of S by y_i. We still assume that the goal is one step away from the start. Keep in mind that the goal can be detected closely behind y_k only when the corresponding ray is visited at distance y_{J_k} again and not earlier! Only the computation of the ratio is different. Altogether, we would like to minimize

$$\sup_k \frac{\sum_{i=1}^{J_k-2} \sqrt{y_i^2 - 2y_i y_{i+1}\cos\frac{2\pi}{m} + y_{i+1}^2}}{y_k} \tag{3.2}$$

which gives a lower bound on (3.1) and on the continuous problem as well. Note that (3.2) still stems from the m-ray version of the problem.

Again, we set $y_{-m+1} = y_{-m+2} = \ldots = y_0 = 1$ for initializing the first visits on the virtual rays, y_{-m+1} stands for the first ray, y_{-m+2} for the second ray and so on. In (3.2), the index k is in the range from $-m+1$ to infinity whereas J_k begins at index 3. In the following, for notational convenience we let S start at index 1. For short, *step j with distance y_j* is also denoted as *step y_j*. Altogether, the following Lemma holds.

LEMMA 3.1. *Let S be a strategy for the discrete m-ray version of the problem. The ratio of (3.2) is never greater than the ratio of (3.1). Minimizing (3.2) gives a lower bound on the discrete m-ray and the continuous version of the problem.*

The simplification above results in the following much more simple interpretation of the problem that reflects (3.2). Figure 4 shows an example for $m = 5$ and with up to 11 steps $y_i := \beta_i x_i$ in the original m-ray version. For (3.2) we can assume that there are only two rays with angle $\frac{2\pi}{m}$ between them and the agent moves successively from one to the other, see Figure 5(i). For every visit the agent only imaginarily decides which ray, l, of the original m rays it would have visit in the m-ray setting, the important indices J_k still determine the ratio. Remember that the ray visited with depth y_k is imaginarily visited the next time at index J_k. For initialization, the index J_{-m+1} represents the first visit on ray 1, the index J_{-m+2} the first visit on ray 2 and so on. For example, in Figure 5(i) $J_{-4} = 1$ and $J_1 = 7$ means that the first of the $m = 5$ rays was visited at y_1 and the next time at y_7 as induced by Figure 4.

In the following the simple interpretation will be denoted as the *$\frac{2\pi}{m}$-ray version*. Any strategy for the *$\frac{2\pi}{m}$-ray version* is uniquely determined by m, by the sequence $S = (y_1, y_2, \ldots)$ and the *visiting order* $J_S = (J_{-m+1}, J_{-m+2}, \ldots)$. Every pair (S, J_S) determines the ratio (3.2). The following lemma holds.

LEMMA 3.2. *Any strategy for the discrete m-ray version of the problem induces a sequence $S = (y_1, y_2, \ldots)$ and a visiting order $J_S = (J_{-m+1}, J_{-m+2}, \ldots)$, the corresponding ratio (3.2) for the $\frac{2\pi}{m}$-ray version of the problem is never greater than the ratio of (3.1). Minimizing (3.2) gives a lower bound on the discrete m-ray and the continuous version of the problem.*

From now on we consider $\frac{2\pi}{m}$-ray version of the problem. Let us assume that an optimal sequence $S = (y_1, y_2, \ldots)$ together with a visiting order $J_S = (J_{-m+1}, J_{-m+2}, \ldots)$ that minimizes (3.2) in the $\frac{2\pi}{m}$-ray version is given. Before step y_i with $i \geq 1$ starts, the m rays have been *virtually* visited up to a certain depth. The target is at least one step away from the origin, so the starting *depth* on every ray is 1. So before step y_i is applied there is a m-vector $D_{i-1} := [y_{i-1_1}, y_{i-1_2}, \ldots, y_{i-1_m}]$ that represents the current depth on the rays in increasing order which means $y_{i-1_j} \leq y_{i-1_{j+1}}$ for $j = 1, \ldots, m-1$. For initialization we have $D_0 = [1, 1, \ldots, 1, 1]$. The first step $y_1 > 1$ yields $D_1 = [1, 1, \ldots, 1, y_1]$. For short let $D_{i-1}[j] := y_{i-1_j}$, the j-th smallest depth among m depth entries.

For an optimal sequence S in the $\frac{2\pi}{m}$-ray version we can choose a *visiting order* (more precisely, we improve the subscript k in the indices J_k) in such a way that at every step y_i the ray with the smallest current depth in D_{i-1} is visited next, which means, that we obtain D_i by deleting $D_{i-1}[1]$ from and inserting y_i into D_{i-1}. Note that y_i is not necessarily inserted into the first place of D_i and $D_{i-1}[2]$ might become $D_i[1]$. If two rays have exactly the same smallest current depth, we visit the ray which was visited earlier.

Intuitively, the above rule will keep the ratio (3.2) as small as possible regardless how the original visiting order was. The numerator of (3.2) is the same for all $i = J_k$ but we can choose k or y_k, respectively. If we do not choose the minimal y_k at step y_i, it *has* to be choosen later at $j = J_k > i$, the numerator increases. The above rule will be denoted as the *smallest-current-depth* visiting order. We will prove that the smallest-current-depth visiting order chooses the current smallest depth immediately and also maximize it for the future, this is optimal.

For example, in Figure 5(i) an optimal visiting order for the first five steps $y_1, \ldots, y_5$ should visit the five *imaginary* rays successively since the starting depth on every ray was 1, this gives $J_{-4} = 1$, $J_{-3} = 2$, $J_{-2} = 3$, $J_{-1} = 4$ and $J_0 = 5$. For step y_6 we set $J_4 = 6$ since y_4 is the smallest current depth on all rays now. This means that y_6 visits the same ray as y_4. Further on we obtain $J_2 = 7$ since y_2 is the smallest current depth at step y_7, then we have $J_6 = 8$, $J_1 = 9$, $J_5 = 10$, $J_9 = 11$ and so on. This is the best visiting order for S.

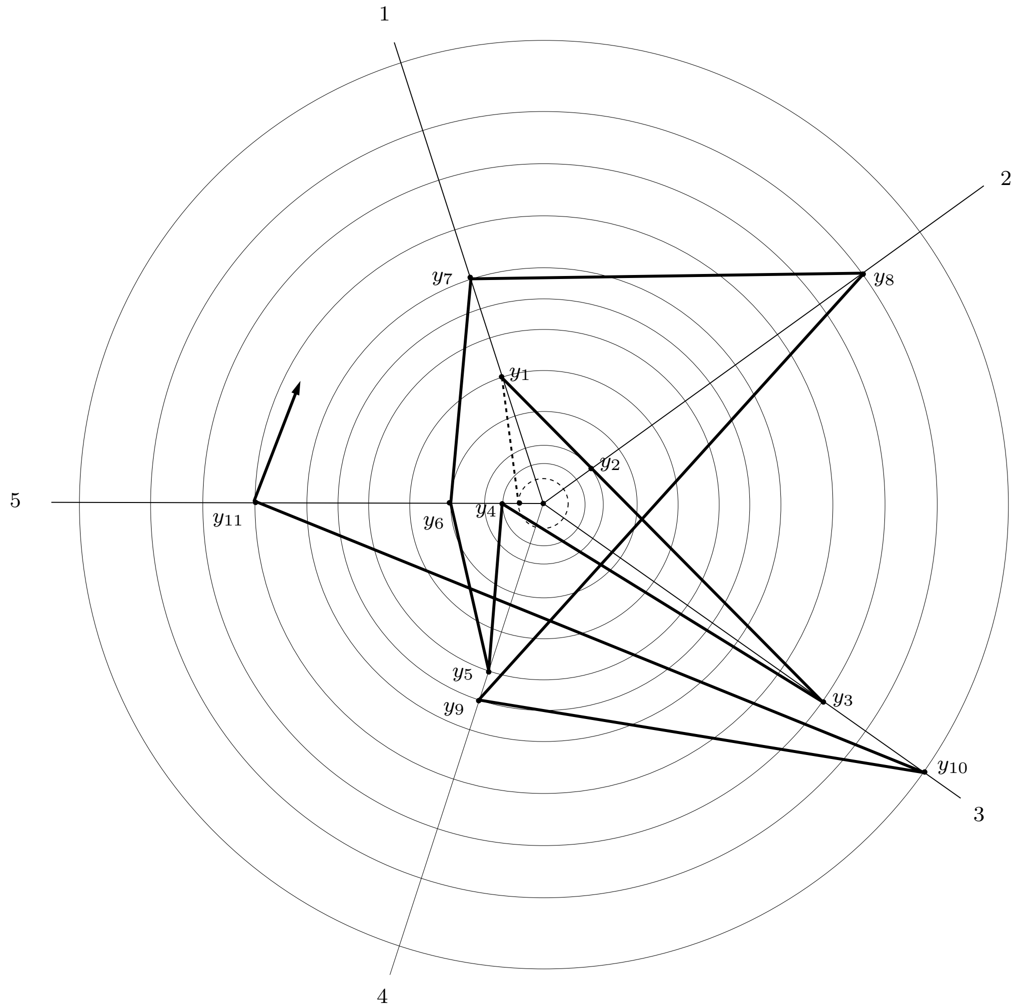
LEMMA 3.3. *Let $S = (y_1, y_2, \ldots)$ be the sequence of a strategy for the $\frac{2\pi}{m}$-ray version of the problem. The smallest-current-depth visiting order minimizes (3.2) among all visiting orders.*

Proof. We did not change the movement of the strategy S. Thus the numerator of ratio (3.2) is the same at *step* y_i with index $i = J_k$ (the sum runs up to index $i - 1$), but we can choose k, that is the denominator y_k among the current depth vector D_{i-1}. At every step y_i the smallest depth in D_{i-1}, $D_{i-1}[1]$, is responsible for the greatest ratio of some future steps.

We show that the smallest-current-depth rule maximizes $D_{i-1}[1]$ for all $i \geq 1$ among all existing visiting orders D_{i-1}. Thus the smallest-current-depth visiting order chooses the current smallest depth immediately and also maximize it for the future, this is optimal.

We show even a bit more. Let $\overline{D}_{i-1}$ denote the depth vector for another visiting rule with the same sequence S. By induction, we prove that $D_{i-1}[j] \geq \overline{D}_{i-1}[j]$ for $j = 1, \ldots, m$ for all $i \geq 1$.

For $i = 1$ this is obviously true, the depth vector is



Induced visiting order: $J_{-4} = 1, J_{-3} = 2, J_{-2} = 3, J_{-1} = 5, J_0 = 4, J_1 = 7, J_2 = 8, J_3 = 10, J_4 = 6, J_5 = 9, J_6 = 11, \ldots$

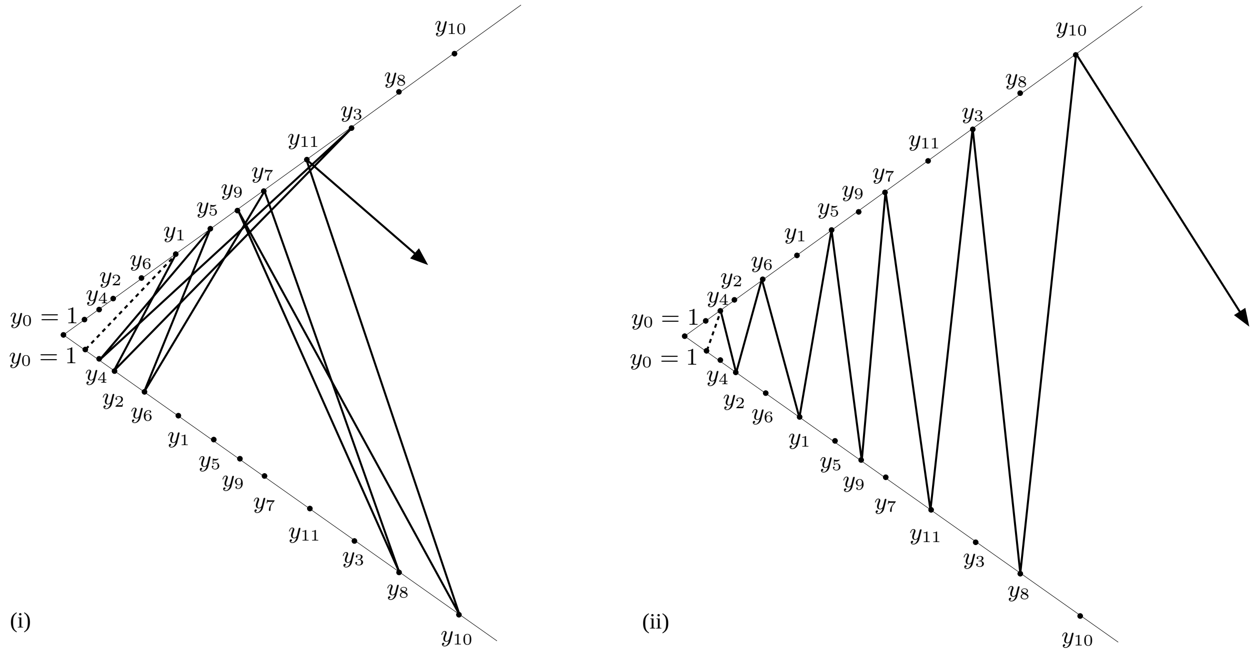
Figure 4: The *original* visits (with $y_i := \beta_i x_i$) of a strategy for $m = 5$ rays gives the induced visiting order $J_{-4} = 1, J_{-3} = 2, J_{-2} = 3, J_{-1} = 5, J_0 = 4, J_1 = 7, J_2 = 8, J_3 = 10, J_4 = 6, J_5 = 9, J_6 = 11, \ldots$ for the $\frac{2\pi}{m}$-ray version of the problem in Figure 5.

initialized by $D_0 = [1, \ldots, 1]$ for every visting order. Let us assume that the smallest-current-depth rule maximizes $D_{i-1}[j]$ up to index $i \geq 1$ for $j = 1, \ldots, m$ among all existing visiting orders. Let $\overline{D}_{i-1}$ denote the ordered depth vector of another visiting order for the same sequence S. The next entry for $\overline{D}_{i-1}$ and D_{i-1} in S is y_i and both orders can replace exactly one entry in $\overline{D}_{i-1}$ and D_{i-1}, respectively. By assumption the smallest-current-depth rule has maximized all entries $D_{i-1}[j]$, that is $D_{i-1}[j] \geq \overline{D}_{i-1}[j]$ for $j = 1, \ldots, m$.

We consider two cases for y_i. In the first case y_i is not greater than any element in D_{i-1}. Thus it is needless to update D_{i-1}, we skip this step and can further improve the ratio for the smallest-current-depth rule. We have $D_{i-1}[j] \geq y_i$ for $j = 1, \ldots m$ and $D_{i-1} = D_i$. If it is needless to insert y_i into $\overline{D}_i$ also, we have $\overline{D}_{i-1} = \overline{D}_i$ and the statement holds for index i. On the other hand, let us assume that we can replace one entry in $\overline{D}_{i-1}$ by y_i. We assume that $\overline{D}_i[l] = y_i$ and that $\overline{D}_{i-1}[k]$ was removed. Obviously we have, $k \leq l$ because of the order in $\overline{D}_i$ and $\overline{D}_{i-1}$. This means $\overline{D}_i[j] = \overline{D}_{i-1}[j]$ for $j = l+1, \ldots m$ and $\overline{D}_i[j] \leq y_i$ for $j = 1, \ldots l$. Altogether, we have $D_i[j] \geq \overline{D}_i[j]$ for $j = 1, \ldots, m$. The statement holds for index i also.

In the second case let y_i be greater than at least one element in D_{i-1}, we delete $D_{i-1}[1]$ and insert y_i for getting D_i. Assuming that $D_i[n] = y_i$, we have $D_i[j] = D_{i-1}[j+1]$ for $j = 1, \ldots, n-1$ and $D_i[j] = D_{i-1}[j]$ for $j = n+1, \ldots, m$. Since y_i is greater than one element in D_{i-1} and $D_{i-1}[j] \geq \overline{D}_{i-1}[j]$ for $j = 1, \ldots, m$, Induced visiting order: $J_{-4} = 1, J_{-3} = 2, J_{-2} = 3, J_{-1} = 5, J_0 = 4, J_1 = 7, J_2 = 8, J_3 = 10, J_4 = 6, J_5 = 9, J_6 = 11, \ldots$

Optimal visiting order: $J_{-4} = 1, J_{-3} = 2, J_{-2} = 3, J_{-1} = 4, J_0 = 5, J_4 = 6, J_2 = 7, J_6 = 8, J_1 = 9, J_5 = 10, J_9 = 11, \ldots$



$y'_1 = y_4, y'_2 = y_2, y'_3 = y_6, y'_4 = y_1, y'_5 = y_5, y'_6 = y_9,$
$y'_7 = y_7, y'_8 = y_{11}, y'_9 = y_3, y'_{10} = y_8, y'_{11} = y_{10}, \ldots$

Optimal visiting order: $J'_i = i + m$

Figure 5: (i) For ratio (3.2) we interpret the situation by successive visits on two rays of angle $\frac{2\pi}{m}$ and with imaginary visits. The visiting order of the corresponding $m = 5$ rays is induced by Figure 4. The optimal smallest-current-depth visiting order for S is given by $J_{-4} = 1, J_{-3} = 2, J_{-2} = 3, J_{-1} = 4, J_0 = 5, J_4 = 6, J_2 = 7, J_6 = 8, J_1 = 9, J_5 = 10, J_9 = 11, \ldots$ (ii) Reordering of S into S' and application of the smallest-current-depth visiting order for S' gives $J'_i = i + m$. The strategy S' is periodic and monotone, can also be applied to Figure 4 and is never worse than S wrt.(3.2).

the entry y_i replaces one entry in $\overline{D}_{i-1}$ also. Let us assume $\overline{D}_i[l] = y_i$ and that $\overline{D}_{i-1}[k]$ was removed. As above we have $k \leq l$. From the properties above we conclude $n \leq l$, otherwise $\overline{D}_{i-1}[l+1] > D_{i-1}[l+1]$, a contradiction to the induction hypothesis.

We conclude $D_i[j] \geq \overline{D}_i[j]$ for $j = l+1, \ldots, m$, these values did not change from $\overline{D}_{i-1}$ to $\overline{D}_i$ and from D_{i-1} to D_i. From $y_i = D_i[n] = D_i[l]$ and $n \leq l$ we conclude, $D_i[j] \geq \overline{D}_i[j]$ for $j = n, \ldots, l$. Finally, we have $D_i[j] = D_{i-1}[j+1]$ for $j = 1, \ldots n-1$. In $\overline{D}_i$ the k-th element was removed with $k \leq l$. Therefore we have $\overline{D}_i[j] = \overline{D}_{i-1}[j]$ for $j = 1, \ldots, k-1$ and $\overline{D}_i[j] = \overline{D}_{i-1}[j+1]$ for $j = k, \ldots l-1$. For $k \geq n$ and for $k < n$ we have $D_i[j] \geq \overline{D}_i[j]$ for $j = 1, \ldots, n-1$.

Altogether, the statement holds for index i also and in general by induction. The smallest-current-depth visiting order maximizes $D_i[1]$, minimizes (3.2) and is optimal among all visiting orders for a given sequence S.

There is also another obvious and simple improvement of a sequence S under the smallest-current-depth visiting order. If y_i does not exceed any entry in D_{i-1} we simply skip this step and further improve (3.2). Note that Lemma 3.3 is a powerful instrument, it also holds for classical m-ray search and some variants, for example on bounded distances.

Let us now assume that the smallest-current-depth visiting order has been applied to S and the depth vector D_{i-1} is updated at every step y_i as mentioned above. We can now easily prove that the m greatest values of $S_{m+i} := (y_1, y_2, \ldots, y_{m+i})$ determine the current depth vector D_{m+i}.

LEMMA 3.4. *Let $S = (y_1, y_2, \ldots)$ be a sequence and let $J_S = (J_{-m+1}, J_{-m+2}, \ldots)$ be the smallest-current-depth visiting order of a strategy for the $\frac{2\pi}{m}$-ray version. The m greatest values in $S_{m+i} = (y_1, y_2, \ldots, y_{m+i})$ determine the current depth vector D_{m+i}. From D_{m+i-1} to D_{m+i} the i-th greatest value of S_{m+i-1}, $D_{m+i-1}[1]$, is deleted and the value y_{m+i} is inserted.*

Proof. The statement holds for $i = 0$ and S_m, the first m values are distributed among m rays, an entry 1 is deleted and the value $y_m > 1$ is inserted. Let us assume that the statement holds for S_{m+i-1}. That is D_{m+i-1} contains the m greatest values of $(y_1, y_2, y_3, \ldots, y_{m+i-1})$ and $D_{m+i-1}[1]$ is the i-th greatest element.

At step y_{m+i} from D_{m+i-1} to D_{m+i} the smallest value $D_{m+i-1}[1]$ is deleted and y_{m+i} is inserted somewhere for getting D_{m+i}. Therefore y_{m+i} is one of the m greatest values in $S_{m+i} = (y_1, y_2, \ldots, y_{m+i})$ and the vector D_{m+i} contains them. Additionally, $D_{m+i}[1]$ is the $(i+1)$-th greatest element of S_{m+i} now. By induction the statement holds.

Of course, after a step with $y_i < y_{i-1}$ the value y_i might become the smallest current depth on all rays and exchanges $D_{i-1}[1]$. As already mentioned we have $y_i > D_{i-1}[1]$ otherwise this step is omitted. Then by our visiting order the corresponding ray should be visited again in the next step y_{i+1}. For technical reasons we simply allow to (*imaginarily*) visit the same ray in two successive steps. By construction this cannot happen *physically* in the original m-ray version. This additional freedom can only improve the ratio (3.2). Any reasonable strategy for the $\frac{2\pi}{m}$-ray will apparently not choose the same ray at two successive steps and this will be shown below.

The next idea is that at the end we really visit the two rays in an *increasing* order. That is, now for a strategy (S, J_S) with smallest-current-depth visiting order, the sequence S will be rearranged to S'.

In the example of Figure 5(i) the strategy $S = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11} \ldots)$ now will be reordered to $S' = (y'_1, y'_2, y'_3, \ldots)$ with $y'_1 = y_4$, $y'_2 = y_2$, $y'_3 = y_6$, $y'_4 = y_1$, $y'_5 = y_5$, $y'_6 = y_9$, $y'_7 = y_7$, $y'_8 = y_{11}$, $y'_9 = y_3$, $y'_{10} = y_8$, $y'_{11} = y_{10}$ and so on in Figure 5(ii). For S' we can again use a *visiting order*, $J'_{S'}$, induced by the smallest current depth. This can not increase the ratio for S' as shown in Lemma 3.3. Obviously, this means that the m rays are now visited with increasing distance and therefore, by the smallest-current-depth rule, in successive order. For index n in S', the index J'_n is exactly $n+m$. Thus, S' is monotone and periodic and the ratio is given by

$$\sup_n \frac{\sum_{i=1}^{n+m-2} \sqrt{{y'_i}^2 - 2y'_i y_{i+1} \cos \frac{2\pi}{m} + {y'_{i+1}}^2}}{y'_n}. \tag{3.3}$$

LEMMA 3.5. *Let $S' = (y'_1, y'_2, y'_3, \ldots)$ be a sequence with entries in increasing order. If $J'_{S'} = (J'_{-m+1}, J'_{-m+2}, \ldots)$ is the smallest-current-depth visiting order of S', the strategy $(S', J'_{S'})$ is monotone and periodic and the ratio is given by (3.3).*

We would like to show that S' is not worse than the original strategy S with respect to the competitive ratio. The ratio (3.2) for index k (or depth y_k) on S is attained at the next visit at index J_k. At index $i = J_k$, y_k represents the smallest current depth. In S' the distance y_k might be visited in another step than in S, that is $y'_n = y_k$ and $n \neq k$. For index n in S', the index J'_n equals exactly $n+m$ as already seen. We would like to compare the ratios for y_k in S and $y'_n = y_k$ in S'. We will show that for the sequence S also $J_k = n+m$ has to be fulfilled, since the visit order in S was induced by current smallest depth. Note that the sum in the numerator of (3.2) and (3.3) still goes to $J_k - 2$ or $J'_n - 2$, respectively, these indices are the same.

For example, in the sequence S' in Figure 5(ii), the ray of $y'_3 = y_6$ is visited next at index 8, which is given by $J'_3 = 8 = 3+m$. In the sequence S in Figure 5(i) the ray of y_6 was also visited again at index 8, represented by $J_6 = 8$. Thus, for comparing the ratios for the same denominator $y'_3 = y_6$ we have to use the sums in the numerator of the ratios of S and S' up to the same index $J_6 - 2 = 6 = J'_3 - 2$. But there might be some different elements since S was reordered in S'. For the differing elements in S and S' up to index $8 = J_6 = J'_3$, we see that *any* of these elements from S, namely y_3, y_8, is strictly greater than any corresponding new element in S', namely $y'_6 = y_9, y'_8 = y_{11}$. We will prove that there can be at most $m-1$ differing values in S' and S up to index $J'_n = J_k$ for $y'_n = y_k$. The above statements hold in general.

LEMMA 3.6. *Let (S, J_S) be an optimal strategy that uses the smallest-current-depth visiting order. Let $S' = (y'_1, y'_2, \ldots)$ be the strategy with entries of S in increasing order and let $J'_{S'} = (J'_{-m+1}, J'_{-m+2}, \ldots)$ be the smallest-current-depth visiting order for S'.*

For the ratio (3.2) for index k and depth y_k for S and ratio (3.3) for S' for index n with depth $y'_n = y_k$ we have $n + m = J_k = J'_n$.

The vectors S_{m+i} and S'_{m+i} differ in at most $j \leq m-1$ values. All the differing values in S_{m+i} are greater than the greatest value of S'_{m+i}.

Proof. We show that for every $i \geq 1$ the i-th greatest element of $S_{i+m-1} = (y_1, y_2, \ldots, y_{i+m-1})$ and the i-th greatest element of $S'_{i+m-1} = (y'_1, y'_2, \ldots, y'_{n+m-1})$ are the same. Let D denote the depth vector of S and let D' denote the depth vector of S'. At step y_{i+m} by Lemma 3.4 we know that in D_{i+m-1} the value y_{i+m} is inserted for the i-th greatest value of $(y_1, y_2, \ldots, y_{i+m-1})$ which is $D_{i+m-1}[1]$. This also means that all values y_j for $j = i+m, i+m+1, \ldots$ are greater than $D_{i+m-1}[1]$ and $D_{i+m-1}[1]$ is the i-th greatest value of S in total, which is y'_i in S' and S'_{i+m-1}.

This holds for every $i \geq 1$. Therefore S_{i+m-1} and S'_{i+m-1} have at least i entries in common. Let us assume that for $j \leq m-1$ the values $y_{i_1}, y_{i_2}, \ldots, y_{i_j}$ of S_{i+m-1} are not in S'_{i+m-1}. Any of the last j values of S'_{i+m-1} is smaller than any value $y_{i_1}, y_{i_2}, \ldots, y_{i_j}$. Otherwise, because S' is the sorted version of S, one of the corresponding values have to be part of S'_{i+m-1}.

Finally, we consider an entry $y'_n = y_k$. From Lemma 3.5 we know that $J'_n = n+m$ holds and y'_{n+m} replaces the n-th greatest element y'_n of $(y'_1, y'_2, \ldots, y'_{n+m-1})$ from D'_{n+m-1} to D'_{n+m}. From Lemma 3.4 we know that y_{n+m} replaces the n-th greatest element of $(y_1, y_2, \ldots, y_{n+m-1})$ from D_{n+m-1} to D_{n+m} also. From above we conclude that the n-th greatest element of $(y_1, y_2, \ldots, y_{n+m-1})$ is exactly y_k which means $J_k = n+m$.

The remaining task is to compare the length of the path (i.e., the numerator of the ratios (3.2) and (3.3)) induced by S_{n+m} and S'_{n+m}, respectively, up to the same index $J'_n - 2 = J_k - 2$ for $y'_n = y_k$. For this comparison we consider simple shortest path problems on two rays. First, we consider the case where S_{n+m} and S'_{n+m} have exactly the same elements. Finally, we take the up to $m-1$ differing elements of S_{n+m} and S'_{n+m} into account.

Let a sequence of distances $S = (y_0, y_1, \ldots, y_n)$ and two rays with opening angle $< \pi$ be given. For convenience, we denote a point with distance y_i from the origin by y_i also. We use images of the corresponding points on both rays as depicted in Figure 5. The task is, to compute a shortest path that starts at the smallest distance y_j in S and visits exactly one of the two images for every distance y_i but has to change sucessively from one ray to the other. We refer to this problem as the *two-ray-shortest-path* problem.

We would like to show that the shortest path has to visit the rays in an increasing order.

Lemma 3.7. *Let $S = (y_1, y_2, \ldots, y_n)$ be a sequence of distances for the two-ray-shortest-path problem. The shortest path which starts at the point with smallest distance from the origin of the rays has to visit the rays in an increasing order.*

The proof is given in the appendix and is shown by triangle inequality and induction. Note that this is somewhat analogous to four point conditions in matrices see [8] and [18]. The above lemma states that the path length induced by S'_{n+m} is never greater than the path length induced by S_{n+m}, provided that they have exactly the same entries.

Unfortunately, S_{n+m} and S'_{n+m} might have $m-1$ different elements as stated above. Fortunately, any of these elements in S_{n+m} is greater than any corresponding element in S'_{n+m}. We compare the two-rays-shortest path Π_S for S_{n+m} and $\Pi_{S'}$ of S'_{n+m}, which means that they visit the same sequence of elements up to y'_i with $i \geq n+1$. The subchain of $\Pi_{S'}$ that visits $(y'_i, y'_{i+1}, \ldots, y'_{n+m})$ is compared to the subchain of Π_S which starts at y'_i also, but has greater elements. In Lemma 3.8 we show that this part of Π_S is greater than the corresponding part of $\Pi_{S'}$, the proof is given in the appendix. Altogether, the path length of S'_{n+m} is never greater than the path length of S_{n+m} even if $m-1$ elements differ.

Lemma 3.8. *Let $S = (y_0, y_1, \ldots, y_n)$ and $S' = (y'_0, y'_1, \ldots, y'_n)$, be two sequences given in increasing order. Let $y_0 = y'_0$ and let $y_j > y'_i$ for $j = 1, \ldots, n$ and $i = 1, \ldots, n$, The two-ray-shortest-path for S' is always strictly smaller than the two-ray-shortest-path for S.*

The above lemmas settle the above problem, the path length induced by S'_{n+m} is never greater than the path length induced by S_{n+m} even if $m-1$ elements differ. The sequence S' is optimal for the $\frac{2\pi}{m}$-ray version of the problem because for every $y'_n = y_k$ the ratio (3.3) is not greater than the ratio (3.2).

Now, we have shown that for the $\frac{2\pi}{m}$-ray version of the problem there is always an optimal strategy $(S', J'_{S'})$ which is monotone and periodic. Fortunately, $(S', J'_{S'})$ is periodic and we can apply this strategy to the original m-ray version as well. But we cannot guarantee to minimize the starting ratio (3.1) exactly because we omitted the last step. But we can guarantee to minimize the following ratio by a periodic and monotone strategy.

$$(3.1') \qquad \sup_k \frac{1}{\beta_k x_k} \cdot \left(\sum_{i=1}^{J_k-2} \beta_i x_i - x_i + \sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos \gamma_{i,i+1} + x_{i+1}^2} \right)$$

Theorem 3.1. *For the original m-ray version of the problem for minimizing (3.1') there is always an optimal periodic and monotone strategy S with $\beta_i = 1$ for all i. The optimal solution gives a lower bound for the continuous version.*

Finally, an optimal solution for (3.1') can be computed by optimizing (3.3) and this gives a lower bound for the continuous version of the search game. For increasing m the lower bound will rise up to the ratio of the logarithmic spiral.

We compute an optimal sequence S' for ratio (3.3). Fortunately, S' is periodic and monotone and fulfills some other nice properties (for example unimodality of the functional) so that a general framework of Gal can be applied, see [15, 14, 1]. We repeat this framework at the end of the appendix and show that it is applicable to (3.3). Thus for discrete m the ratio (3.3) is minimized by an exponential sequence $y_i' = a^i$. Simple arithmetic shows that the ratio is given by $f(a, m) = \frac{a^{m-1}}{a-1}\sqrt{1 - 2a\cos\frac{2\pi}{m} + a^2}$. Thus, we can analytically find the value $a_{\min}$ that minimizes $f(a, m)$. For increasing m the corresponding value $f(a_{\min}, m)$ converges to 17.289... For example, for $m = 100000$ we compute $a_{\min} = 1.000009764\ldots$ and $f(a_{\min}, 100000) = 17.289\ldots$

THEOREM 3.2. *Spiral search is optimal.*

4 Conclusion and Future Work

In this paper we show that the logarithmic spiral conjecture is true for the searching-for-a-point-in-the-plane problem and this settles an old open fundamental search problem introduced by Gal [14].

The main open question is whether we can prove the spiral conjecture for the searching-for-a-line-in-the-plane problem also, see [14, 11]. In this setting the smallest-current-depth visiting order means that we should extend the convex hull of the strategy at its smallest radius. One can be optimistic that an approach similar to the one presented here will succeed.

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5 Appendix

First, we would like to prove that the strategy S' adapted from S in Section 3 always has a smaller path length. Only the last at most $m-1$ elements differ from S to S'. The differing elements in S are all greater than the greatest element of S'. The following lemma considers the case for identical entries.

Let a sequence of distances $S = (y_0, y_1, \ldots, y_n)$ and two rays with opening angle $< \pi$ be given. For convenience, we denote a point with distance y_i from the origin by y_i also. We use images of the corresponding points on both rays as depicted in Figure 6. The task is, to compute a shortest path that starts at the smallest distance y_j in S and visits exactly one of the two images for every distance y_i but has to change sucessively from one ray to the other. We refer to this problem as the *two-ray-shortest-path* problem.

LEMMA 3.7. *Let $S = (y_1, y_2, \ldots, y_n)$ be a sequence of distances for the two-ray-shortest-path problem. The shortest path which starts at the point with smallest distance from the origin of the rays has to visit the rays in an increasing order.*

Proof. The statement is shown by induction on the length of S. For two elements in S this is trivial. So we consider a sequence S with $n+1$ elements and assume that the statement holds for all sequences with less than n elements. If the shortest path for $n+1$ elements starts with the segment of smallest slope, we are done for the following reason. We delete the smallest element, y_j, out of S and the shortest path starting at the second smallest value, y_i, visits the images in an increasing order by induction hypothesis. Therefore we can combine the segment with smallest slope, $y_j y_i$, with the shortest path starting at y_i. Thus, we obtain an overall shortest path that visits the images in increasing order.

So let us assume that the shortest path starting for $n+1$ images does not start with the segment, $y_j y_i$, of smallest slope, see Figure 6(i). Therefore the path starts with a segment $y_j y_l$. The ongoing path has to visit y_i. If it already ends at y_i, we replace $y_j y_l$ by $y_j y_i$ which is smaller and we are done. If it does not end at y_i, we have a path, $P_{y_l}^{y_i}$, from y_l to y_i and an ongoing shortest path, $P^{\geq y_i}$ with at least one segment. Let $y_i y_k$ denote the first segment of $P^{\geq y_i}$. We will show that the overall path is not optimal by constructing a shorter path that starts with $y_j y_i$, see Figure 6(ii). We start with $y_j y_i$ and then we move backwards along $P_{y_l}^{y_i}$ and visit y_l. From y_l

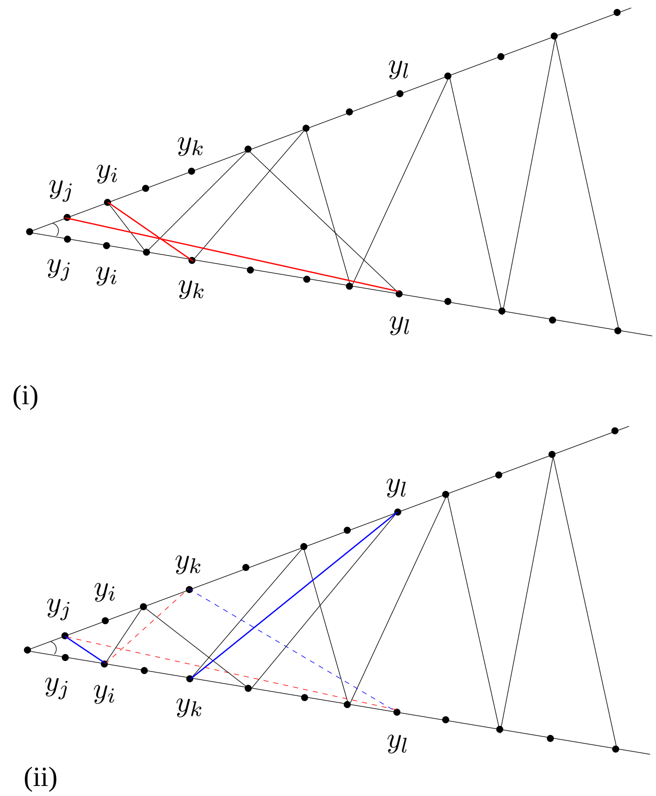

Figure 6: The shortest path problem for a sequence of values and their images. (i) If the shortest path does not start with $y_j y_i$, we can replace $y_j y_l$ and $y_i y_k$ by $y_j y_i$ and $y_l y_k$. (ii) After rearrangement, triangle inequality shows that the path has to start with $y_j y_i$. Induction shows that the shortest path starting at y_i has to visit the images in increasing order.

we choose a segment $y_l y_k$ and proceed with $P^{\geq y_k}$. This is a path that starts with $y_j y_i$ and visits all images. We have replaced $y_j y_l$ and $y_i y_k$ by $y_j y_i$ and $y_l y_k$. Since we can arrange the situation, see Figure 6(ii), so that $y_j y_l$ and $y_i y_k$ cross each other by triangle inequality the *new* path is strictly smaller. A contradiction to the assumption, the shortest path starts with the segment $y_j y_i$.

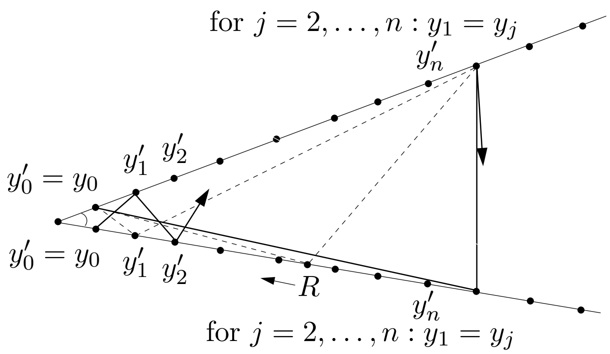

Figure 7: Comparing the shortest path of two sequences $S = (y_0, y_1, \ldots, y_n)$ and $S' = (y'_0, y'_1, \ldots, y'_n)$, with $y_0 = y'_0$ and $y_j > y'_i$ for $j = 1, \ldots, n$ and $i = 1, \ldots, n$. The path of sequence S is as small as possible if $y_j = y_1$ for $j = 2, \ldots, n$ holds. In this case we replace $y_0 y_1$ and $y_1 y_2$ by the shorter sequence $y_0 y'_1$ and $y'_1 y_2$. Starting from $R = y_1$ and moving R towards y'_0 the path length $y_0 R$ and $R y_2$ decrease until a specular reflection occurs, then it increases until $R = y'_0$. By induction the path of S is always greater.

The following lemma considers the last $m - 1$ differing entries of S and S'.

Lemma 3.8. *Let $S = (y_0, y_1, \ldots, y_n)$ and $S' = (y'_0, y'_1, \ldots, y'_n)$, be two sequences given in increasing order. Let $y_0 = y'_0$ and let $y_j > y'_i$ for $j = 1, \ldots, n$ and $i = 1, \ldots, n$, The two-ray-shortest-path for S' is always strictly smaller than the two-ray-shortest-path for S.*

Proof. For $n = 1$ the statement holds. From $y'_1 < y_1$ we conclude that $y'_0 y'_1$ is smaller than $y_0 y_1$. Now, we assume that the statement holds for $n - 1 \geq 1$ and two sequences $S = (y_0, y_1, \ldots, y_n)$ and $S' = (y'_0, y'_1, \ldots, y'_n)$ with the given property are given. A worst-case situation for S' occurs if all y_j for $j = 1, \ldots, n$ are nearly the same. This makes S as small as possible, since S has to be visited in increasing order by Lemma 3.7. Therefore we assume that $y_j = y_1$ for $j = 2, \ldots, n$. The optimal path for S starts at $y_0 < y_1$ with a segment $y_0 y_1$ and then moves successively from one ray to the other at distance y_1. We replace the first two steps for S, namely $y_0 y_1$ and $y_1 y_2$ by $y_0 y'_1$ and $y'_1 y_2$.

The length of $y_0 y'_1$ plus the length of $y'_1 y_2$ is always smaller than the length of $y_0 y_1$ plus the length of $y_1 y_2$ by the following argument. Starting at $R = y_1$ we *move* the reflection R to the left, see Figure 7. In the beginning the length of $y_0 R$ and $R y_2$ decrease until R represents a specular reflection. If a the specular reflection is achieved and we continue to move R to the left the length $y_0 R$ and $R y_2$ increases again. From now on the maximal increase for $y_0 R$ and $R y_2$ occurs if R equals y_0. Comparing $y_0 y_0$ and $y_1 y_1$ gives the result. The length of $y_0 y'_1$ plus the length of $y'_1 y_2$ is always smaller than the length of $y_0 y_1$ plus the length of $y_1 y_2$.

Therefore we can improve the path for S if we simply replace y_1 by y'_1. Obviously, by induction hypothesis we can now apply the assumption for $S = (y_1, \ldots, y_n)$ and $S' = (y'_1, \ldots, y'_n)$. Although S was shortened it is still greater than S'.

The rest of the appendix is dedicated to the optimization of functionals due to the framework of Gal, see [15, 14, 1]. We would like to minimize the supremum of

$$(3.3)\quad \frac{\sum_{i=1}^{n+m-2} \sqrt{{x_i}^2 - 2 x_i x_{i+1} \cos \frac{2\pi}{m} + {x_{i+1}}^2}}{x_n} =: F_{n+m-1}(x_1, x_2, \ldots, x_{n+m-1})\,.$$

The given problem results in an optimization problem for functionals $F_k(X)$ with sequences $X = (x_1, x_2, x_3, \ldots)$. For two sequences $X = (x_1, x_2, x_3, \ldots)$ and $Y = (y_1, y_2, y_3, \ldots)$ let $X + Y := (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots)$ and $A \cdot X := (A \cdot x_1, A \cdot x_2, A \cdot x_3, \ldots)$ for a constant A.

Theorem 5.1. *(adapted from Gal [15, 14], Alpern and Gal [1], and Schuierer [26])*
Given a sequence of functional $F_k(X)$ for all $k \geq k_0$ and sequences $X = (x_1, x_2, x_3, \ldots)$ and $Y = (y_1, y_2, y_3, \ldots)$ with $x_i > 0$ and $y_i > 0$.
If the following conditions hold for F_k:

(i) *F_k is continuous,*

(ii) *F_k is* unimodal, *which means: $F_k(A \cdot X) = F_k(X)$ and $F_k(X + Y) \leq \max\{F_k(X), F_k(Y)\}$,*

(iii) $\liminf_{a \mapsto \infty} F_k \left(\frac{1}{a^k}, \frac{1}{a^{k-1}}, \ldots, \frac{1}{a}, 1 \right) = \liminf_{\epsilon_k, \epsilon_{k-1}, \ldots, \epsilon_1 \mapsto 0} F_k \left(\epsilon_k, \epsilon_{k-1}, \ldots, \epsilon_1, 1 \right),$

(iv) $\liminf_{a \mapsto 0} F_k \left(1, a, a^2, \ldots, a^k \right) = \liminf_{\epsilon_k, \epsilon_{k-1}, \ldots, \epsilon_1 \mapsto 0} F_k \left(1, \epsilon_1, \epsilon_2, \ldots, \epsilon_k, \right),$

(v) $F_{k+1}(x_1, \ldots, x_{k+2}) \geq F_k(x_2, \ldots, x_{k+2})$.

then

$$\sup_k F_k(X) \geq \inf_a \sup_k F_k(A_a)$$

with $A_a = a^0, a^1, a^2, \ldots$ *and* $a > 0$. *The supremum of the functional is minimized by an exponential function.*

Note that the given form of condition (v) is a replacement shown by Schuierer [26], see also Alpern and Gal [1]. Obviously, the functional of (3.3) fulfills condition (i), (iii), (iv), (v) and the part $F_k(A \cdot X) = F_k(X)$ of (ii). The remaining task is to prove that unimodality holds in the additive sense which is $F_k(X+Y) \leq \max\{F_k(X), F_k(Y)\}$. Normally, this is the most difficult thing, here it is easy.

Let $\theta \in (0, \pi/4]$, we can show that

$$\sqrt{x_i^2 + -2\cos(\theta) x_i x_{i+1} + x_{i+1}^2} \; + \sqrt{y_i^2 - 2\cos(\theta) y_i y_{i+1} + y_{i+1}^2} \; \geq \sqrt{(x_i{+}y_i)^2 - 2\cos(\theta)(x_i{+}y_i)(x_{i+1}{+}y_{i+1}) + (x_{i+1}{+}y_{i+1})^2}$$

holds. This is the triangle inequality of the vectors

$$A := \left((x_i - x_{i+1}) \cos\frac{\theta}{2}, (x_i + x_{i+1}) \sin\frac{\theta}{2} \right)$$

and

$$B := \left((y_i - y_{i+1}) \cos\frac{\theta}{2}, (y_i + y_{i+1}) \sin\frac{\theta}{2} \right)$$

for the Euclidean norm, which is $||A+B|| \leq ||A|| + ||B||$.

Now let $F_k(X) \leq K$ and $F_k(Y) \leq K$, which is also true for $K := \max\{F_k(X), F_k(Y)\}$. We have

$$\sum_{i=1}^{k} \sqrt{x_i^2 - 2\cos(\theta) x_i x_{i+1} + x_{i+1}^2} \; + \sum_{i=1}^{k} \sqrt{y_i^2 - 2\cos(\theta) y_i y_{i+1} + y_{i+1}^2} \; \leq K \cdot (x_{k-m+2} + y_{k-m+2})$$

and we can prove $F_k(X+Y) \leq K$ by the inequality shown above. This gives unimodality in the additive sense for (3.3).